SPECTRAL SYNTHESIS IN BANACH MODULES

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Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
May, 1976

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ACKNOWLEDGEMENTS

The author is indebted to the people who in some way contributed to the completion of this thesis.

I express my heartfelt appreciation to my parents for fostering a desire to learn and for their encouragement throughout the years. I am thankful to my wife, Ersilia, and daughter, Angela, for time, understanding and meaning.

My gratitude is extended to the educators with whom I have learned, especially those who have shared experiences and enthusiasm in mathematics. In particular, for his direction and many suggestions which have made this thesis possible, my profoundest appreciation goes to Dr. James T. Burnham.

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CHAPTER I

INTRODUCTION AND PRELIMINARIES

Algebra has had a profound affect in analysis, particularly visible in harmonic analysis. One instance is in operatory theory where I. Kaplansky originated the use of module theory in his investigations [66]. The first extensive work on modules in analysis, specifically on Banach modules, was Rieffel's study [89]. Subsequent application of the Banach module concept is evident in the present day literature. A naturally occurring phenomena in analysis, the Banach module concept was evident in prior investigations, but only infrequent references were made and no extensive advantage of the module perspective has been exploited.

Such has been the case in the theory of spectral synthesis. Indeed, Domar [30] has recognized the value of the module concept with regard to this problem. Kitchen [69] and Dunkl [34] both worked in a module setting, but not to the extent of employing the concept of "spectrum" as Domar had done. It is our intention to regard spectral synthesis questions in a Banach module setting and take advantage of this approach. In particular, we contend that the Banach module context is conducive for spectral synthesis considerations, and indeed provide a "proper setting" for such questions. We attempt to unify existing theories and establish a spectral synthesis theory for Banach modules in the spirit of the weak-star spectral synthesis problem as initiated by Beurling, Godement, et al.

Our primary goal is to show that not only can a spectral synthesis problem be formulated for Banach modules, but that a theory evolving from such basic questions renders a theory analogous to that known for commutative Banach algebras. In the process of determining this interrelation, we introduce a duality condition for Banach modules termed "bi-annihilation invariance" which renders an interesting posture to our considerations.

Our design is to provide a minimum background of results and historical development of spectral synthesis for an appreciation of our intentions. We intend to accomplish this in Chapters I and II. Chapter I furnishes basic definitions and fundamental results required for the text while Chapter II is a brief exposition of spectral synthesis intended for one to obtain a "flavor" of the subject. The main body of the thesis is comprised in Chapters III to V. Chapter III is essential as it renews the concept of spectrum and its fundamental properties in the module context. We also supply an "examples section." Furthermore, problems related to spectra are formulated and the forementioned condition of bi-annihilation invariance is introduced. We conclude Chapter III with structural questions regarding this duality condition.

While the thesis is partly expository, the impact of our investigation is present in Chapters IV and V. In particular, the development of a spectral synthesis theory in Banach modules is carried out. Chapter IV includes elementary results in spectral synthesis for modules which are necessary for a fruitful theory. Chapters IV and V both give evidence of the validity of our contention. Stemming from Chapters III and IV, Chapter V presents a Banach module formulation of a Wiener-Ditkin-Shilov theorem which utilizes standard yet significant techniques extended to Banach modules. Our contribution concludes with an application of spectra to almost periodicity. It becomes apparent that "bi-annihilation invariance" is significant to almost periodic considerations as well as our Banach module perspective.

Chapter VI concludes the thesis indicating problems for further research. We indicate some open problems arising from our study. In addition, certain difficulties are pointed out to convey the inherent troubles in formulating and proposing a spectral synthesis theory for Banach modules.

§1. Conventions and Notation

The remainder of the chapter is devoted to providing definitions and basic facts essential to an understanding of the subsequent chapters. We set forth conventions, terminology and notations to be used throughout the sequel. Furthermore, our standard references are for harmonic analysis, Rudin [96, Chapters 1-2], Commutative Banach algebra theory, Loomis [81], and for functional analysis Dunford and Schwartz [35]. The reader well-versed in these areas may proceed to Chapter II or III after section 1.1.

The sets of complex numbers and real numbers will be denoted by $\mathbb{C}$ and $\mathbb{R}$, respectively. The additive group of integers will be denoted by Z and the set of complex numbers of modulus one by T, the latter also identified with the unit circle as a multiplicative group. We also regard $\mathbb{R}$ as an additive group.

The symbol G denotes a locally compact abelian group (LCAG), and $\hat{G}$ its dual group with Haar measures written as m, dm, dm(x), or dx (respectively, $\hat{m}$, $d\hat{m}$, $d\hat{m}(\hat{x})$ or $d\hat{x}$). Elements of G are written in lower case letters x, y, $\cdots$ while elements of the dual group will "carry a hat," such as $\hat{x}$, $\hat{y}$, $\cdots$. Since our primary interest is in a commutative theory, the group operation will be written as addition.

Whenever E is a Banach space, E^* will denote the dual or conjugate space (i.e., the space of all continuous linear functionals on E, generally not the algebraic dual). We consider only complex Banach spaces. The duality between E and E^* is written as a pairing: if $x \in E$ and $x^* \in E$, $(x,x^*) \equiv x^*(x)$.

The symbols $\langle x_n \rangle$ and $\langle x_\alpha \rangle$ denote a sequence and net, respectively. No reference to the index set will be made unless required for emphasis. For a topological space (X,τ) and $S \subset X$, we use

(a) $S^c = X \setminus S \equiv$ compliment of S in X;

(b) $c\ell_X^\tau(S) \equiv \tau$-closure of S in X ($c\ell(S)$ if the topology and space are clear from context);

(c) $\mathrm{int}(S) \equiv$ interior of S;

(d) $\partial_X(S) \equiv$ boundary of S in X.

In addition, the following spaces will be encountered frequently.

1. $L^p(G)$, the space of (Borel) measurable functions such that

$$||f||_p \overset{\text{def.}}{\equiv} [\int_G |f(x)|^p dm(x)]^{1/p} < \infty, \; 1 \leq p < \infty,$$

and the space of all bounded (Borel) functions $L^\infty(G)$ normed by

$$||f||_\infty \overset{\text{def.}}{\equiv} \text{ess sup}|f(x)| \qquad\qquad p = \infty.$$

Our definitions here are rather imprecise, for detailed considerations see Reiter [92] and Hewitt and Ross [60, 264].

2. The important subspaces of continuous functions:

(a) $C(G) \equiv \{f : G \to \mathbb{C} | f$ is continuous$\}$;

(b) $C_u(G) \equiv \{f \in C(G) | f$ is bounded and uniformly continuous$\}$;

(c) $C_o(G) \equiv \{f \in C(G) | f$ vanishes at infinity$\}$;

(d) $C_c(G) \equiv \{f \in C(G) | f$ has compact support$\}$;

all with the supremum norm $||\cdot||_\infty$. Evidently, we have $C(G) \supseteq C_u(G) \supseteq C_o(G) \supseteq C_c(G)$, and they all coincide if G is compact.

§2. Banach Algebras

A complex algebra A is a commutative Banach algebra if it is commutative and a Banach space with norm $||\cdot||_A$ satisfying

$$||a_1a_2||_A \leq ||a_1||_A||a_2||_A \quad \text{for all} \quad a_1, a_2 \in A.$$

The symbol "A" will denote a commutative Banach algebra <u>without identity</u> unless something to the contrary is stated. However, we allow the concept of "approximate identity." A net $\langle e_\alpha \rangle$ is an <u>approximate identity</u> for A if for each $a \in A$, we have $||ae_\alpha - a||_A \to o$. It is <u>bounded</u> in case $||e_\alpha|| \leq d$ for all e_α where $d > o$ is constant.

The set of all complex homomorphisms on A is denoted by $\Delta(A)$. Each element of $\Delta(A)$ is necessarily bounded (continuous). For each $a \in A$, $\hat{a}$ denotes an associated function on $\Delta(A)$ defined by the relation $\hat{a}(h) = h(a)$ for all $h \in \Delta(A)$. Clearly, $\Delta(A) \subseteq A^*$, and for each $a \in A$, $\hat{a}$ is obtained by restricting the functional $a^{**} : A^* \to \mathbb{C}$ to $\Delta(A)$, where a^{**} is given by $(a^*, a^{**}) = (a, a^*)$ for all $a \in A$. The map $a \to \hat{a}$ is a homomorphism called the <u>Gelfand transform</u> of A. This entails $\widehat{a_1a_2} = \hat{a}_1\hat{a}_2$ for all $a_1, a_2 \in A$. We let $\hat{A}$ be the set $\{\hat{a} : a \in A\}$, the so-called <u>Gelfand representation</u> of A. The <u>cospectrum</u> (<u>zero set</u>) of $a \in A$ is the set $\text{cosp}(a) \equiv \{\hat{x} \in \Delta(A) : \hat{a}(\hat{x}) = o\}$. We denote the <u>support</u> of $\hat{a}$, $a \in A$, by $\sigma(a) \equiv c\ell\{\hat{x} \in \Delta(A) : \hat{a}(\hat{x}) \neq o\}$.

Ideals

An ideal I in A is regular if A/I has an identity. If I is closed, A/I is a Banach algebra with respect to the quotient norm, in particular, if I is regular and maximal, $A/I \cong \mathbb{C}$ (isometrically isomorphic). The set of all regular maximal ideals of A is in one-to-one correspondence with the elements of $\Delta(A)$. If $\bigcap_{M\in\Delta(A)} M = \{0\}$, or equivalently, the Gelfand transform is one-to-one, then A is called semisimple. The Gelfand topology imposed on $\Delta(A)$ is the weak topology on $\Delta(A)$ induced by the functions in $\hat{A}$. This coincides with the relative topology on $\Delta(A)$ as a subset of A^* provided with the weak-star topology (induced by A). The space $\Delta(A)$ is then a locally compact Hausdorff space called the maximal ideal space of A. We now define some special ideals. Let E be a closed subset of $\Delta(A)$. We define the following:

(a) $I(E) \equiv \{a \in A : \hat{a}$ vanishes on $E\}$, the kernel of E;

(b) $J_o(E) \equiv \{a \in A : \sigma(a)$ is compact and disjoint from $E\}$;

(c) $J(E) \equiv c\ell[J_o(E)]$;

(d) $A_c \equiv \{a \in A : \sigma(a)$ is compact$\}$.

We write $\hat{x}$ for an element in $\Delta(A)$ which corresponds to the closed maximal ideal $I_{\hat{x}} \equiv I(\{\hat{x}\})$. The hull of an ideal I in A, hull(I), is the set $\{\hat{x} \in \Delta(A) : \hat{a}(\hat{x}) = o$ for all $a \in I\}$. We have hull$(I) = \bigcap_{a \in I} \operatorname{cosp}(a)$ and so the hull(I) is a closed subset of $\Delta(A)$. Clearly, the hull operation reverses inclusion. Observe that $I(E)$ and $J(E)$ are both closed ideals, and in fact, $I(E)$ is the largest closed ideal with hull equal to E while $J(E)$ is the smallest such ideal.

We also have that $I(E)$ is the intersection of all the regular maximal ideals in E and $\mathrm{hull}(I)$ is the set of all regular maximal ideals which include I.

The maximal ideal space $\Delta(A)$ can also be provided with the "hull-kernel" topology, that is, with the topology defined by the closure operation: $\mathrm{hull}(I(E)) = \overline{E}$, $E \subseteq \Delta(A)$. In the event that the hull-kernel topology coincides with the Gelfand topology, we say that A is a regular Banach algebra. A regular Banach algebra is then a commutative Banach algebra in which $\hat{A}$ is a regular function algebra.

Important Banach Algebras

1. The Group Algebra $L^1(G)$. We define convolution on $L^1(G)$ as follows: for $f, g \in L^1(G)$, the convolution of f with g, $f*g$, is the number (possibly infinite)

$$\int_G f(x)g(y - x)dx, \; y \in G.$$

The space $L^1(G)$ with convolution as multiplication is a regular semi-simple Banach algebra. The Gelfand transform coincides with the Fourier transform and the ideal of compactly supported Fourier transforms is dense in $L^1(G)$.

2. The Measure Algebra. The space consisting of all bounded regular Borel measures on G is denoted by $\mathfrak{M}(G)$. With respect to convolution (of measures) defined by

$$\mu*\lambda(S) = \int_G \lambda(S - x)d\mu(x),$$

and norm $||\mu||_M \equiv |\mu|(G)$ (the total variation of μ), $\mathfrak{M}(G)$ is a Banach algebra, perhaps the most important of all Banach convolution algebras, and certainly the most mysterious. The Fourier-Stieltjes transform of $\mu \in \mathfrak{M}(G)$ is denoted by $\hat{\mu}$ and given by

$$\hat{\mu}(\hat{x}) = \int_G (-x,\hat{x})\,d\mu(x)$$
$$= \int_G \overline{(x,\hat{x})}\,d\mu(x) \qquad \hat{x} \in G.$$

The Fourier-Stieltjes transform is <u>not</u> the Gelfand transform, but we denote by $\hat{S}$, the set of images of the Fourier-Stieltjes transform of the elements in $S \subseteq \mathfrak{M}(G)$. In fact, the F-S transform is the restriction of the Gelfand transform to $\hat{G}$.

By the Radon-Nikodym theorem, $L^1(G)$ is identified as a subalgebra of $\mathfrak{M}(G)$. We mention that $\mathfrak{M}(G)$ has an identity, namely the point mass concentrated at zero. Among other peculiar properties of $\mathfrak{M}(G)$ are the following: $\mathfrak{M}(G)$ is <u>not</u> regular, non-symmetric (the set of Gelfand transforms is not closed under complex conjugation), $\hat{G}$ is an open subset of $\Delta(\mathfrak{M}(G))$, and the absolutely continuous measures in $\mathfrak{M}(G)$ vanish outside $\hat{G}$. We refer the reader to Rudin [96] for the cited properties of $\mathfrak{M}(G)$.

§3. Fundamental Results in the Theory of Commutative Banach Algebras

In addition to the essential definitions in the theory of Banach algebras discussed in section §2, we state the following facts without proof, taken from Loomis [81], see also Hewett and Ross [61, chapter 10].

<u>Fact 1</u>. If A is a regular Banach algebra and $\hat{x} \in \Delta(A)$, then there is an $a \in A$ such that $\hat{a} \equiv 1$ in some nbhd. of $\hat{x}$.

Fact 2. If A is a regular Banach algebra and $E \subset \Delta(A)$ a compact set, then there is an $a \in A$ such that $\hat{a} \equiv 1$ on E.

Fact 3. If A is a regular Banach algebra, E a compact subset of $\Delta(A)$, F a closed subset of $\Delta(A)$ and $E \cap F = \emptyset$, then there is an $a \in A$ such that $\hat{a} \equiv 0$ on F and $\hat{a} \equiv 1$ on E. (Fact 3 will be indispensable in our subsequent work.)

Fact 4. (Abstract Wiener Theorem). Let A be a regular semisimple Banach algebra with A_c dense in A, then every proper closed ideal is contained in a regular maximal ideal.

Spectral Synthesis

Let A be a semisimple regular Banach algebra. A closed subset of E of $\Delta(A)$ is a set of spectral synthesis if there exists a unique closed ideal with hull E.

We now state a well-known characterization of sets of spectral synthesis.

Fact 5. Let A be a semisimple regular Banach algebra and E a closed subset of $\Delta(A)$, the following are equivalent:

(i) E is a set of spectral synthesis,

(ii) $J(E) = I(E)$,

(iii) if I is a closed ideal with $\text{hull}(I) = E$, then $I = I(\text{hull}(I))$.

A well-known example in which every closed subset of the maximal ideal space is a set of spectral synthesis is $C(X)$, X a compact Hausdorf space. However, this is a rarity and we, therefore, state sufficient conditions for sets to be of spectral synthesis.

Local Membership and Wiener-Ditkin Condition

Let I be an ideal in a semisimple regular Banach algebra. An element $a \varepsilon A$ belongs locally to I at $\hat{x} \varepsilon \Delta(A)$ if there exists an $a_1 \varepsilon I$ such that $\hat{a} \equiv \hat{a}_1$ on some nbhd. of $\hat{x}$. If $\hat{x}$ is the point at infinity, $\hat{a} \equiv \hat{a}_1$ outside some compact set.

Fact 6. Let A be a semisimple regular Banach algebra and I an ideal in A. If $a \varepsilon A$ belongs locally to I at each point of $\Delta(A)$ (and at the point at infinity if $\Delta(A)$ is non-compact), then $a \varepsilon I$.

Fact 7. Let A be a semisimple regular Banach algebra and I an ideal in A. An element $a \varepsilon A$ belongs locally to I at each point in $\mathrm{int}(\mathrm{hull}(I))$ and at each point not in $\mathrm{hull}(I)$.

Let A be a semisimple regular Banach algebra. We say A satisfies condition (D) if for each $\hat{x} \varepsilon \Delta(A)$ and $a \varepsilon I_{\hat{x}}$, there exists a sequence $\langle a_n \rangle \subset A$ such that $\hat{a}_n \equiv 0$ on a nbhd. U_n of $\hat{x}$ and $||aa_n - a||_A \to 0$. If $\Delta(A)$ is non-compact, the condition must also be satisfied at the point at infinity.

We now state the "best" conditions known to insure spectral synthesis.

Fact 8 (Wiener-Ditkin-Shilov Theorem). Let A be a semisimple regular Banach algebra satisfying condition (D) and I be a closed ideal in A. If $a \varepsilon A$ satisfies $\mathrm{hull}(I) \subseteq \mathrm{cosp}(a)$ and $\partial\mathrm{cosp}(a) \cap \mathrm{hull}(I)$ contains no non-empty perfect subset, then $a \varepsilon I$. In particular, if $\partial\mathrm{hull}(I)$ contains no non-empty perfect subsets, then $\mathrm{hull}(I)$ is a set of spectral synthesis.

§4. Spectral Synthesis in $L^1(G)$

We now state a few important results in spectral synthesis for $L^1(G)$. In particular, the results of section §3 apply since $L^1(G)$ is a semisimple, regular, commutative Banach algebra and $(L^1(G))_c$ is dense in $L^1(G)$. We refer the reader to Rudin [96, Chapter 7] for details and further exposition.

Fact 9. The closed translation invariant subspaces of $L^1(G)$ are the same as the closed ideals of $L^1(G)$. This is not true for non-closed ideals.

Fact 10. Let G be a compact abelian group, then spectral synthesis obtains for $L^1(G)$, i.e., if I is a closed ideal of $L^1(G)$ and $f \in L^1(G)$ with $\mathrm{hull}(I) \subset \mathrm{cosp}(f)$, then $f \in I$.

Fact 11 (Wiener Tauberian Theorem). If $f \in L^1(G)$, then the closed translation invariant subspace generated by f is all of $L^1(G)$ if and only if $\hat{f}$ is zero free. Note: Fact 4 is equivalent to this for $A = L^1(G)$.

A semigroup S of G is called angular if $0 \in c\ell(\mathrm{int}\,(S))$.

Fact 12. Angular semigroups of $\hat{G}$ are sets of spectral synthesis. (See Hille and Phillips [62, p. 265], and de Leeuw and Mirkil [23, pp. 361-362]). Our next fact is a famed negative result concerning non-synthesis.

Fact 13 (Malliavin). Let G be non-compact, then $\hat{G}$ contains a closed subset that is not of spectral synthesis.

The first example in this spirit is due to L. Schwartz [97]. He exhibited that $E = \{x \in \mathbb{R}^n : |x| = 1\}$, $n \geq 3$, is not a set of spectral synthesis. However, for the case $n = 2$, Herz [59] showed that the circle is a set of spectral synthesis.

Fact 14 (Helson). If I_1 and I_2 are closed ideals of $L^1(G)$ such that $I_1 \subsetneq I_2$ and $\mathrm{hull}(I_1) = \mathrm{hull}(I_2)$, then there exists a closed ideal I such that $I_1 \subsetneq I \subsetneq I_2$. Furthermore, there are $\mathfrak{c}$ such ideals.

As mentioned in the introduction, we shall expound on the history of spectral synthesis in Chapter II.

§5. Banach Modules

Let A be a commutative Banach algebra. Define a Banach A-module B as a Banach space $(B, ||\cdot||_B)$ such that

(i) B is an algebraic A-module with respect to an operation $*_B$;

(ii) there exists a constant $C > 0$ such that

$$||a*_Bb||_B \leq C \; ||a||_A||b||_B \quad \text{for all} \quad a \in A,\ b \in B.$$

We actually consider left-Banach A-modules, but we suppress the side of operation (of course, we could consider right-modules as well). We have defined Banach A-modules with respect to commutative Banach algebras because those are of primary interest to us. The letter B will be reserved for Banach A-modules.

A submodule M of B is a closed linear subspace of B such that $a*_Bb \in M$ for all $a \in A$ and $b \in M$. The smallest norm-closed submodule generated by $b \in B$ will be denoted by $[b]$. If τ is another topology on B, $\overline{[b]}^{\tau}$ will denote $c\ell^{\tau}(A*_Bb)$. In particular, $[b] = c\ell(A*_Bb)$. If B_1 and B_2 are Banach A-modules with $B_2 \subseteq B_1$, then for $M \subseteq B_2$:

$$c\ell^{\tau}_{B_1}(M) \equiv \tau\text{-closure of } M \text{ in } B_1.$$

We now state some ways in which Banach modules may be obtained.

1. Any Banach space is a Banach $\mathbb{C}$-module with respect to scalar multiplication.

2. If I is a closed ideal of A, I is a Banach A-module with respect to the algebra operation.

3. If M is a closed submodule of a Banach A-module B, then $(B/M, ||\cdot||_{B/M})$ is a Banach A-module ($||\cdot||_{B/M}$ is the quotient norm).

4. If B is a Banach A-module, B^* is a Banach A-module with respect to the operation $\circledast$ defined by the relation

$$(b, a \circledast b^*) = (a*_B b, b^*) \quad a \in A, b \in B, b^* \in B^*.$$

Hereafter when B^* is regarded as an A-module, it is understood to be with respect to this operation. In particular, if $B = A$, A^* is regarded as an A-module with respect to "$\circledast$".

The <u>order submodule</u> of B is the submodule $\{b \in B : a*_B b = 0\}$. A module B is <u>order-free</u> if the order submodule is trivial. The <u>essential part</u> of B, denoted B_e, is the submodule $A*_B B$. A Banach A-module B is called <u>essential</u> if $A*_B B$ is (norm-) dense in B.

We also have the notion of "approximate identity" as for Banach algebras. A Banach A-module B has an <u>approximate identity</u> $\langle e_\alpha \rangle$ in A if $||e_\alpha *_B b - b||_B \to 0$ for each $b \in B$. We say $\langle e_\alpha \rangle$ is <u>bounded</u> if there exists a constant $d > 0$ such that $||e_\alpha||_A \leq d$ for all e_α.

The Banach space of all continuous module homomorphisms from a Banach A-module B_1 to a Banach A-module B_2 will be denoted by $\mathrm{Hom}_A(B_1, B_2)$. Thus, T is in $\mathrm{Hom}_A(B_1, B_2)$ if and only if T is a continuous homomorphism from $B_1 \to B_2$, in particular,

$$T(a*_{B_1} b) = a*_{B_2} T(b) \quad \text{for all} \quad a \in A,\ b \in B_1.$$

We denote the multiplication operators on A and B by

$$T_a : B \to B, \quad T_b : A \to B \quad \text{where} \quad T_a(b) = a*_B b = T_b(a).$$

Hence, we have $T_a \in \mathrm{Hom}_A(B,B)$ and $T_b \in \mathrm{Hom}_A(A,B)$.

We now end this section with one of the most important results in the theory of Banach modules.

Fact 15 (Hewitt-Curtis-Figà-Talamanca). Let A be a commutative Banach algebra with a bounded approximate identity and B a Banach A-module. The following are equivalent:

(i) B is essential;

(ii) B has a bounded approximate identity;

(iii) for $b_o \in B$, there is an element $b \in B$ and an $a \in A$ such that $b_o = a*_B b$.

For extensive results in the theory of Banach A-modules, see Hewitt and Ross [61, p. 263 ff], Rieffel [89], Comisky [21], [22], and A. W. Graven [50] as well as [51], [52], [53], and [54].

§6. Topologies on Banach Modules

It will be evident from the discussion in Chapter II that we will have a need for a topology weaker than the norm topology on a Banach A-module B. We present two topologies of primary interest in our investigation.

The General Strict Topology

The first topology that we consider is a generalization of R. C.

Buck's "strict topology" on $C(X)$, X a locally compact Hausdorff space (see Buck [13]). The results presented here regarding the general strict topology are due to Sentilles and Taylor [100]. We also refer the reader to Shapiro [101].

The general strict topology on a Banach A-module B is the topology induced by the seminorm $b \to ||a*_B b||_B$ $(a \in A)$. This means that $\{p_a : a \in A\}$, $p_a(b) \overset{\text{def}}{\equiv} ||a*_B b||_B$, generates a locally convex topology on B for which p_a is continuous for each $a \in A$. We assume the topology is Hausdorff (in this connection see Remark 3.0 (5) in III §1). In particular, $\langle b_\alpha \rangle \subset B$ converges to b in the (general) strict topology if and only if $||a*_B b_\alpha - a*_B b||_B \to 0$ for every $a \in A$. We write $\langle b_\alpha \rangle$ <u>β-converges to b</u> to mean $\langle b_\alpha \rangle$ converges to b in the (general) strict topology.

For each $b \in B$, the operator T_b on A is in $\mathrm{Hom}_A(A,B)$ (recall §5) so that if we impose the uniform operator topology on B via the norm $||b|| \overset{\text{def.}}{\equiv} \sup\limits_{||a|| \leq 1} ||a*_B b||$, B can be regarded as a "subset" of $\mathrm{Hom}_A(A,B)$ and the β-topology is the restriction of the operator topology, i.e., $b_\alpha \to b$ in the β-topology if and only if $T_{b_\alpha} \to T_b$ in $\mathrm{Hom}_A(A,B)$. We shall denote the uniform operator topology as the σ-topology. Evidently, we have

$$\text{norm convergence} \Longrightarrow \sigma\text{-convergence} \Longrightarrow \beta\text{-convergence}.$$

<u>Fact 1</u>. The collection of all sets $V_a \equiv \{b : ||a*_B b||_B \leq 1\}$, $a \in A$, is a base for the nbhd. system at zero in the β-topology. We now suppose A has a bounded approximate identity $\langle e_\alpha \rangle$.

Fact 2. The following are equivalent:

(i) The strict topology and norm topologies are equivalent:

(ii) $\sup_{||b|| \leq 1} ||e_\alpha *_B b - b||_B \to 0$;

(iii) B is a Frechét space in the strict topology.

Fact 3. $\mathrm{Hom}_A(A,B_e)$ is complete in the σ-topology and contains B_e as a dense subset. In particular, (B,β) is complete if and only if $B = \mathrm{Hom}_A(A,B)$. In this case, $c\ell_B^\beta(A*B) = B$.

For the next result, we refer to Shapiro [101]. The bounded weak-star topology on A^* induced by A is the strongest topology which agrees on bounded sets with the weak-star topology (see Dunford and Schwartz [35, V §5]).

Fact 4. The bounded weak-star topology induced on $L^p(G)$ by $L^q(G)$ $(1 < p \leq \infty, \frac{1}{p} + \frac{1}{q} = 1)$ coincides with the strict topology on $L^p(G)$ as an $L^1(G)$-module if and only if G is compact.

We now make the following observations relating the β-topology and norm topology on submodules.

Let M be a (norm-) closed submodule of B. Then

(i) $c\ell_B^\beta(M) \cap B_e = M$

In particular, for $b \in B_e$, we have $[b] = \overline{[b]}^\beta$.

(ii) The β-limits of elements in B need not be back in B, or if $B \subseteq B_1$, $c\ell_{B_1}^\beta(M) \cap B = c\ell_B^\beta(M)$

(This is a more general relation than (i), where $B_1 = B$ and $B = B_e$ in (i)).

We can see this latter observation by considering the case $A = B = L^1(G)$, $B_1 = \mathfrak{M}(G)$ with the relation $f*g_\alpha \to f*\mu$ for all $f \in L^1(G)$ holding for some $\langle g_\alpha \rangle \subset L^1(G)$, yet $\mu \in \mathfrak{M}(G) \setminus L^1(G)$.

Although we shall not make explicit use of it, we define a related topology for the sake of completeness and its usefulness in studying multipliers. We still assume A has a bounded approximate identity in the following definition and Facts 5-6.

The κ-topology on B is the topology generated by the seminorms $b \to ||e_\alpha *_B b||_B$, $\langle e_\alpha \rangle \subset A$ a bounded approximate identity for A. This topology is locally convex, Hausdorf and finer than the norm topology. Clearly, β-convergence implies κ-convergence.

Fact 5. The κ and β topologies agree on σ-bounded sets.

Fact 6. A net $\langle b_\alpha \rangle \subset B$ β-converges to b if and only if $\langle b_\alpha \rangle$ is σ-bounded and κ-convergent.

We again cite Sentilles and Taylor [102].

A Weak Topology on B

We define the $*$-topology on B in terms of nets. Net convergence is characterized as follows: $\langle b_\alpha \rangle \subset B$ $*$-converges to $b \in B$ if and only if

$$(a *_B b_\alpha, b^*) \to (a *_B b, b^*) \quad \text{for all} \quad a \in A,\ b^* \in B^*.$$

Fact 7. If either B or B^* is essential, the $*$-topology on B coincides with the weak-topology on B induced by B^*.

This follows from the definition of essential and the property of the module operation "$\circledast$" (see 5 (4)). Clearly, β-convergence implies $*$-convergence.

Fact 8. If $B = A^*$, the weak-star topology on B is stronger than the $*$-topology on B, the latter being the weak topology induced by A^{**}.

We refer to Dunford and Schwartz [35, V §3.4] for a discussion of weak topologies. Our $*$-topology is not explicitly presented, but the desired properties are not difficult to ascertain from the results stated in the reference.

CHAPTER II

SPECTRAL SYNTHESIS: AN EXPOSITION

The question of representing a given continuous function on the unit circle by a trigonometric series was an impetus in the development of the theory of Fourier series. This basic question underlies many investigations in the more abstract theories of Fourier analysis and harmonic analysis. It is a matter of deciding whether a function f, say continuous, can be determined from its harmonic components, hence a study of the formula $f(x) = \sum_n \hat{f}(n)e^{inx}$. This is called the problem of the spectral synthesis of f.

We present a brief history of the spectral synthesis problem as a prelude to our considerations in the succeeding chapters. We will expound on the development of the weak-star spectral synthesis of bounded functions and the spectral synthesis problem in the group algebra $L^1(G)$. It is intended that the reader who is not familiar with the subject become exposed to a sketch of the development of this problem. Furthermore, an awareness of the evolution of the spectral synthesis question may serve as a foundation for comprehension of the theory for Banach modules proposed in the subsequent chapters. We lay no claim of completion in this exposition, but intend to establish a historical portrayal of spectral synthesis for a basis of the concepts encountered in the text.

As mentioned in the opening paragraph, the problem of spectral synthesis is concerned with synthesizing a given function if its "spectrum" is known. One may view this as determining closed translation invariant subspaces in certain Banach spaces by means of the characters they contain since these characters and spectra of the given function coincide in the classical case.[1] Indeed, the perspective via translation invariant subspaces was foremost in the work of N. Wiener and his Tauberian results [108] although not stated as such. For instance, Wiener's famed result is that for an integrable function on $\mathbb{R}$, the translates of the function span $L^1(\mathbb{R})$ if and only if it has a zero-free Fourier transform (see Chapter I §4, Fact 4). Prior to Wiener, explicit references to translation invariant subspaces were not made. An example is the work of Weyl and Peter which emphasizes the Fourier expansion for continuous functions on a compact Lie group.[2] I. Segal working with $G = \mathbb{R}$ and W. Rudin working with G discrete infinite abelian have shown that Wiener's theorem does not hold for all $L^1(G)$, $1 < p < 2$. However, for $f \in L^p(G)$, $1 \leq p < \infty$, if the co-spectrum of f is "simple enough," the closed translation invariant subspace generated by f may be equal to $L^p(G)$ (see for example the conditions in R. E. Edwards [39] for an arbitrary locally compact abelian group G, and H. Pollard [88] and C. Herz [60] for $G \equiv \mathbb{R}$).

Arne Beurling [8] considered the problem of spectral synthesis of bounded functions as follows: Given a function $\varphi \in C_u(\mathbb{R})$, is φ in

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[1] See R. E. Edwards [37, Chapter 11], [36, Chapter 2].

[2] We refer to Hewitt and Ross [61, §40–§42]. The exposition presented in the "notes" to Chapter 10 contains an excellent historical view and we do not with intent infringe on that work.

the "narrow-closure" of the span $\{e_n : n \in sp(\varphi),\ e_n(x) = e^{inx}$ for $x \in \mathbb{R}\}$, where the "spectrum of φ," $sp(\varphi)$, is the set of characters contained in the narrow closure of the translation invariant subspace generated by φ? He had introduced the "narrow" topology and exhibited that every non-trivial translation invariant subspace of $C_u(\mathbb{R})$ which is closed in this narrow topology contains a character. Wiener's theorem was also noted to be a corollary to this result and Beurling had defined the "spectrum of φ" as above. It should be noted that the motivation for this area of study originates in the study of integral equations, and the modern treatment is not only due to Beurling, but also to N. Wiener [106] and T. Carleman [19].

With the development of harmonic analysis on locally compact groups and the evolution of the theory of Banach algebras, subsequent studies were made. Roger Godement [48] treated the problem of spectral synthesis for bounded measurable functions on a locally compact abelian group by means of the spectrum defined as a particular subset of the dual group. He defined the spectrum of $\varphi \in L^{\infty}(G)$, $sp(\varphi)$, as the hull of the ideal $\{f \in L^1(G) : f * \varphi = 0\}$ and noted its equivalence to Beurling's definition (incidentally, Godement's definition of spectrum is essential to our considerations in this thesis). Godement also showed Beurling's theorem and Wiener's theorem were, in fact, equivalent. In regard to Godement [49], it is of interest to see Koosis [70] where he discovered and resolved a subtle error. In addition, Beurling [10] reverted to the spectral synthesis question and formulated an alternate, but equivalent, definition of spectrum as $\hat{G} \cap \overline{[\varphi]}^{w^*}$. He also proved the validity of spectral synthesis for a class of weighted algebras.

Beurling [10] stimulated other works in this area. Most notably is the work of Domar [30]. Domar employed Banach algebra theory to investigate the spectral synthesis problem as posed by Beurling. In Domar [31], he abbreviates and simplifies the proof of the theorem in Beurling [8] while in Domar [30] he defined different spectra and determined relationships between them as well as generalized results of Beurling. In fact, Domar worked with certain Banach modules. The theory there encompasses, in part, previous theory as $L^\infty(G)$ can be regarded as a representation space of the group algebra $L^1(G)$. Domar continued his work on the spectral synthesis problem, for example see Domar [34] where he emphasizes the dual aspect and provides applications to function theory. Domar [31], [32] and [33] are pertinent as well. As another important contribution to the theory of spectral synthesis, Herz [58] considers the development of the spectral synthesis issue for $L^\infty(G)$ as well as spectral analysis questions. A well-presented view of the overall problem, it served to bring known results into a concise form.

Although the validity of spectral synthesis for compact groups, for instance $C(T)$, had been known, the general problem for $L^\infty(\mathbb{R})$ with the weak-star topology was still open. In attempting to resolve the spectral synthesis problem, alternate types of spectral synthesis were considered. A concept known as $\mathfrak{A}$-spectral synthesis evolved. This deals with the validity of spectral synthesis in some smaller subspace $\mathfrak{A}$ of the space in question. Typically, it refers to a subalgebra of $L^1(G)$. Beurling and Pollard studied this concept and established the validity of $\mathcal{R}$-spectral synthesis in particular spaces.[3]

With respect to the spectral synthesis of bounded functions as initiated by Beurling [10], we have the equivalent problem for the group algebra $L^1(G)$. The struggle to determine the validity or non-validity of spectral synthesis for $L^\infty(G)$ can be viewed in terms of the validity or non-validity of spectral synthesis for $L^1(G)$. This is due to the fact that a closed set of E of $\hat{G}$ is a "set of spectral synthesis for $L^1(G)$" if and only if E is a "set of spectral synthesis for $L^\infty(G)$." More precisely, the question of spectral synthesis for the group algebra may be posed in terms of ideals as follows: Given a closed subset E of $\hat{G}$, does there exist a unique closed ideal I whose hull is E? A set of spectral synthesis for $L^1(G)$ is any set for which the answer is positive (see I §4). Analogously, a closed set E in $\hat{G}$ is a set of spectral synthesis for $L^\infty(G)$ if there is a unique weak-star closed translation invariant subspace of $L^\infty(G)$ whose spectrum is E. Since these concepts turn out to be equivalent, the spectral synthesis problem can be formulated in a number of ways. We enumerate some as follows (refer to I §2 for notation):

(1) If $\varphi \in L^\infty(G)$ and $f * \varphi = 0$ for every $f \in J(E)$, then is $f * \varphi = 0$ for every $f \in I(E)$?

(2) If $f \in L^1(G)$ and $\hat{f} \equiv 0$ on E, then can f be approximated in $L^1(G)$ by functions $g \in L^1(G)$ such that $\hat{g} \equiv 0$ on some open set containing E?

(3) Determine which closed subsets of $\hat{G}$ are of spectral

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[3]See Hewitt and Ross [61, p. 550] for further results in addition to other off-springs of spectral synthesis.

synthesis for $L^1(G)$.

(4) Determine which closed subsets of $\hat{G}$ are of spectral synthesis for $L^\infty(G)$.

Insofar as the particular spectral synthesis problem for $L^1(G)$ is concerned, it appears as though R. Godement [48] is the first known to explicitly state this problem.[4] However, there were prior references which indicate a concern toward resolution of such a query. For example, Gelfand [45] posed the question of determining which commutative Banach algebras with unit enjoy the property that every closed ideal is the intersection of maximal ideals containing it. In fact, in his work with convolution equations and the class $L^1(\mathrm{IR}) * \mu$ for $\mu \in \mathfrak{M}(\mathrm{IR})$, Beurling [8] solved the spectral synthesis problem for a particular class of spaces, but no explicit mention of the spectral synthesis problem was made.

In 1948, L. Schwartz [98] showed that the unit sphere in IR^3 is not a set of spectral synthesis for $L^1(\mathrm{IR}^3)$. Hence, for $L^1(\mathrm{IR}^3)$ and, in fact, for $L^1(\mathrm{IR}^n)$, $n \geq 3$, spectral synthesis was seen to fail. The general question for locally compact abelian groups remained open. Although more examples of non-synthesis were given, for instance see the citations to Schwartz and Dixmier in [61], the validity or non-validity of spectral synthesis for $L^1(G)$, $G \neq \mathrm{IR}^n$, $n \geq 3$, and non-compact, was not established until 1959 by P. Malliavin. The failure of spectral synthesis for all non-compact locally compact abelian groups was established in (Malliavin's) three papers involving techniques inspired

[4] This is remarked in Hewitt and Ross [61, p. 550], and to the best of the author's knowledge is correct.

by Kahane and Katznelson (for explicit references see Hewitt and Ross [61, p. 602]). In this connection, it is enlightening to see Rudin [96, chapter 7]. The spectral synthesis problem for $L^1(G)$ is therefore seen to be hopelessly difficult, and consequently so is the corresponding problem for $L^\infty(G)$.

In contrast, we mention the relation of almost periodicity to spectral synthesis. In the early history of spectral synthesis, the spectral synthesis of all closed translation invariant subspaces of almost periodic functions on G had been proven by John von Neumann [87] in 1934. The principle result was that if S is a closed translation invariant subspace, then the closure of $S \cap \hat{G}$ is equal to S. The definition of almost periodicity for continuous functions on G was von Neumann's extension to groups of Bochner's definition [11]. Furthermore, Delsarte [24] and L. Schwartz [97] provided another "success story" in the theory of spectral synthesis by means of the theory of mean periodic functions.[5] The main result is that every closed translation invariant subspace S of the "mean periodic functions" contains functions of the form $t \to t^m \exp(\alpha t)$ and is the closure of all such functions. This work has been extended to all locally compact abelian groups by R. J. Elliott [41], [42] and J. E. Gilbert [46] independently. With regard to this we refer to the work of Ehrenphreis cited in [41], [42].

As in the case of most general theories, results in specific spaces paved the way for the more abstract concepts. A spectral synthesis theory in Banach algebras developed after $L^1(G)$ spectral synthesis

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[5] For additional references see [61, p. 551].

considerations evolved. We should point out that Gelfand's initiation of the theory of Banach algebras and Shilov's study of Banach algebras which were regular, commutative, and semisimple provided incentive for the spectral synthesis problem to be investigated in these type of objects.[6] G. W. Mackey [83], Godement [48], Loomis [81], Segal [99], Kaplansky [65], et al. were instrumental in the evolution of spectral synthesis for Banach algebras based on the group algebra case. The problem in Banach algebras emphasizes characterization of closed ideals with respect to cospectra (or hulls, see I §3-4). Sufficient conditions for the validity of spectral synthesis were determined. Most significantly are the conditions of the Wiener-Ditkin-Shilov Theorem (I §3, Fact 8) which precipitated from the overwhelming work of Wiener, Ditkin's results prior to the Gelfand theory (in particular the citation "Ditkin [1]" in [61, p. 731] which contains the heart of the present day formulation of the Wiener-Ditkin-Shilov theorem), and Shilov's contributions (for instance [102, I §4 Theorem 14]).

Insofar as work in Banach modules is concerned, there appears but a few spectral synthesis or spectral analysis considerations in print. Domar [30], [31], [32], [33] exploits a Banach module context to a certain extent, in particular, his study of duals of Banach algebras and group algebra modules. Even though his investigations head in a different direction than the one proposed in the thesis, they are significant and do lay groundwork for various studies. Kitchen [69] uses the module concept to an advantage in his study of almost periodicity which

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[6] We dare not omit mention of Naimark's contribution to the theory of Banach algebras. Indeed, his work [86] was a factor in the foundation of Banach algebras.

involves a spectral synthesis question. However, we note that he does not utilize the idea of spectrum. Another work dealing with spectral synthesis in Banach modules is Dunkl [34]. There is an explicit reference to spectral synthesis questions in unitary modules, but the approach is more functional than spectra-oriented. Apparently no other works in a spectral synthesis theory for Banach modules are available, although specific instances do exist without reference to synthesis in modules.[7] Indeed, all the work on the spectral synthesis of bounded functions can be regarded as spectral synthesis for particular group algebra modules. A noteworthy study is that made by K. de Leeuw and H. Mirkil [23] for the case $C_o(G)$. No module reference is made, but as in the study of Segal algebras, the inherent properties and advantages of Banach module conditions are prominent.[8]

The spectral synthesis problem reveals that a primary difficulty is the topological nature of the sets in question, for instance the boundary of the spectrum of the function to be synthesized as in the Wiener-Ditkin-Shilov theorem. A topological "thinness" entails considerations for various problems which are related to spectral synthesis. Study of so called "thin-sets" is an area of harmonic analysis which accentuates the difficulties involved in recapturing elements from its harmonic

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[7]During the course of preparing this thesis, Kitchen and Robbins have spectral synthesis considerations for a class of Banach modules, but again <u>no</u> use of the notion of spectrum is made. See their preliminary report [70].

[8]For an extensive study of Segal algebras, originally due to Segal [99], see Reiter [92], [93]. For their generalizations see [20], [14], [15], [16], [18]. Moreover, the vast amount of literature on these algebras emphasizes the role played by Banach modules in their study. For instance see [61, p. 263 ff] and [17].

components. It is significant and mandatory to cite the work of T. K. Körner [72], [73], [74] and the collection of works in [80] with respect to the study of those type of sets. We refer to the bibliography of [80] for further explicit references.[9] In addition, N. Th. Varopoulos has contributed to this area in connection with Kronecker sets, Helson sets, and unbounded synthesis.[10] Moreover, Varopoulos' contribution does not end there. He provided a new perspective toward the overall problem of spectral synthesis by utilizing the concept of tensor products in conjunction with Kronecker sets to construct sets of non-synthesis (see [61, p. 749] for references). Varopoulos' idea proved fruitful and is well-recognized in the field of spectral synthesis.

In conclusion, it is to be restated that our survey is not only brief but incomplete. The primary course of our venture into spectral synthesis is to establish a spectral synthesis theory for Banach modules analogous to the spectral synthesis theory for bounded functions. The relationship to the spectral synthesis theory for Banach algebras will be indicated and proven to be similar to the relationship stated for $L^\infty(G)$-spectral synthesis and $L^1(G)$-spectral synthesis in our exposition. Any elimination of significant works is not intended as a shun of that particular contribution, but is merely an oversight due to the limitations of the author.

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[9] Our aim is not to present a detailed account of the development of thin sets, nor their description, but to indicate the existence of such objects relating to spectral synthesis.

[10] We refer the reader to [61, pp. 602-605] for further remarks.

CHAPTER III

SPECTRA IN BANACH MODULES

The objectives of this chapter are to introduce the concept of "spectrum" for Banach modules (§2–§3) and indicate a spectral synthesis theory in the context of Banach modules (§5–§6). This includes the definition of sets of spectral synthesis for modules and associated problems. We also provide an "examples" section (§4) so that the reader may have concrete objects in mind throughout the sequel. In particular, we introduce a duality property for modules which embraces the spirit of spectral synthesis.

§1. Basic Assumptions and Conventions

It is necessary to impose restrictions on the algebras and Banach modules we work with to obtain a "fruitful" theory. Indeed, the conditions will not be stringint but necessary!

Let $(A, ||\cdot||_A)$ be a commutative Banach algebra <u>without identity</u> and $(B, ||\cdot||_B)$ a Banach A-module with respect to an operation $*_B$. We make the following basic assumptions on A and B throughout the remainder of the text, unless stated to the contrary.

<u>A1</u> A is regular and semisimple;

<u>A2</u> A has bounded approximate identities;

<u>A3</u> A_c is dense in A;

<u>A4</u> $A *_B b = 0$ implies $b = 0$;

<u>A5</u> $a *_B B = 0$ implies $a = 0$, i.e., B is order-free. For non-discrete G, the $L^1(G)$-modules $(L^\infty(G))^*$ and $M(G)^*$ are <u>not</u> order-free because $L^\infty(G)$ and $M(G)$ are not essential (see Graven [50, p. 39]).

<u>A6</u> Singletons are of spectral synthesis.

<u>Remarks 3.0</u>

1. Assumptions A1-A3 entail that the Abstract Wiener Theorem obtains for A (I, §3, Fact 4).

2. The approximate identities may be assumed to consist of elements with compactly supported Gelfand transforms by A3.

3. In the event that A has an approximate identity for B, A4 is automatically satisfied, hence this is the case if B is essential.

4. Assumption A5 entails that the general strict topology on B is Hausdorf, or that A "separates" the points (elements) of B.

5. Assumption A6 is satisfied by most examples of interest, and all the examples we consider or encounter in practice shall satisfy this condition. For completeness, we provide an example of an algebra which <u>does</u> <u>not</u> satisfy A6.

<u>Example</u> (Reiter [92, p. 35])

Let $n \geq 3$. A Function f on $\mathbb{R}^n$ is radial if there is a function f^+ defined on $\mathbb{R}_+ \equiv [0,\infty]$ such that $f(x) = f^+(x)$ for $x \in \mathbb{R}^n$. Let $A = \{f \in L^1(\mathbb{R}^n) : f$ coincides a.e. with a radial function$\}$ and have the L^1-norm. The Banach algebra A contains closed primary ideals which are not maximal, i.e., $\{x\}$ is not a set of spectral synthesis, $x > 0$. To see this observe that $I_1 =$

$\{f \in A : \widehat{f^{+}}(x) = \widehat{f^{+'}}(x) = 0\}$ is closed and primary but not maximal. The requirement $x > 0$ is essential since $\{0\}$ is known to be a set of synthesis for $n \geq 2$.

Conventions

In practice, the Banach A-module B may satisfy $A \cap B \neq \phi$. For instance, this is the case for $A = L^1(G)$ and B one of the "usual" Banach A-modules. Of course, $A \cap B$ is to be understood in the sense that A and B are subsets of a larger space E and the intersection is taken in E. On the other hand, $A \cap B$ may be in some natural sense empty. This is the case if $B = C_o(\hat{G})$ and $A = L^1(G)$ with $G \neq \hat{G}$ and with respect to the operation $f*g \overset{\text{def.}}{\equiv} \hat{f} \cdot g$, $f \in L^1(G)$, $g \in C_o(\hat{G})$. In the event that $A \cap B$ is nonempty, we require B to be compatible with A, i.e., if $b \in A \cap B$ and $a \in A$ we require $a *_B b = ab$. To the author's knowledge, Comisky [21, p. 14] was the first to formally recognize the importance of this condition.

The subscript on the module operation "$*_B$" will usually be omitted unless emphasis is desired. The algebra operation will not be denoted, i.e., if $a_1, a_2 \in A$, the product of a_1 and a_2 is $a_1 a_2$. If the algebra has an involution we write "a^*" for the image under the involution, this is not to be confused with an element in A^*. Clarity on this matter is to be obtained from context.

The symbol "R" will denote an algebra and "Q" will always denote an algebraic R-module, hence no topology is to be imposed on R nor Q (observe the difference between "R" and "IR").

As pointed out in the introduction, the nature of this thesis is partly expository, we thus emphasize that some of the results are known. Hence, we shall distinguish between these and so-called "new" results within the body of the text to maintain consistency in transition. Since we are proposing a "Banach-module perspective" of a well-established theory, the reader is urged not to misinterpret the author's intentions in the use of well-known terminology and utilization of standard techniques as a pretense of originality.

§2. The Spectrum

In undertaking a program to develop a spectral synthesis theory for Banach modules, one of the main problems is in the selection of a "suitable" topology. This is, of course, common to both spectral synthesis and spectral analysis. We shall work with some "weak" topology (see I, §6) that affords a "weak-star" spectral synthesis theory for Banach modules in the spirit of Beurling [10], Godement [48], Herz [59], et al. However, the results of this section <u>are not</u> "tied down" to any particular topology on B. The differences to be encountered will be resolved at the appropriate time.

We now begin our development with some basic definitions.

<u>Definition 3.1</u>. Let S be a subset of B. <u>The annihilator ideal of S with respect to A</u> is the set

$$\{a \in A : a*s = 0 \text{ for all } s \in S\}.$$

Let K be a subset of A. <u>The annihilator submodule of K with respect to B</u> is the set

$$\{b \in B : a*b = 0 \text{ for all } a \in K\}.$$

We denote these by $S^{\perp A}$ and $K^{\perp B}$, respectively. Observe that $S^{\perp A}$

(resp. $K^{\perp_B}$) is indeed an ideal of A (resp. submodule of B). In fact, the continuity of the maps $T_b : A \to B$ and $T_a : B \to B$ (see I, §5) together with the relations

$$S^{\perp_A} = \bigcap_{b \in S} T_b^{-1}(0_B) \quad \text{and} \quad K^{\perp_B} = \bigcap_{a \in K} T_a^{-1}(0_B),$$

imply that $S^{\perp_B}$ is closed in A and that $K^{\perp_B}$ is (norm-) closed in B. Moreover, $K^{\perp_B}$ is $*$-closed. To verify the latter assertion, suppose $\langle b_\alpha \rangle \subset K^{\perp_B}$ $*$-converges to $b \in B$. Then $(a*b_\alpha, b') \to (a*b, b')$ for all $a \in A$ and $b' \in B^*$. For $a \in K$, this means that $a*b_\alpha = 0$ for all α and hence $(a*b, b') = 0$ for all $a \in K$, $b' \in B^*$. Now $a*b = 0$ for all $a \in K$ and we have $b \in K^{\perp_B}$. Therefore, $K^{\perp_B}$ is $*$-closed.

For singleton subsets, $\{a\} \subset A$ or $\{b\} \subset B$, we simply write $a^{\perp_B}$ for $\{a\}^{\perp_B}$ and $b^{\perp_A}$ for $\{b\}^{\perp_A}$. We may also at times suppress the subscript on the symbol "$\perp$" if the "annihilator space" is readily identified. In addition, if a succession of annihilator operations is necessary, we write, for example, $S^{\perp_A \perp_B \perp_A}$ for $((S^{\perp_A})^{\perp_B})^{\perp_A}$ if $S \subseteq B$, and similarly for $K \subseteq A$.

We now record some elementary properties of annihilators as a lemma. Recall that by convention, the following requires no topology and is simply an algebraic result.

<u>Lemma 3.1</u>. Let $\mathbf{Q}$ be an R-module. Let I and J be subsets of R, and M and N be subsets of $\mathbf{Q}$, then the following hold:

(i) if $I \subseteq J$ ($M \subseteq N$), then $I^{\perp_{\mathbf{Q}}} \supseteq J^{\perp_{\mathbf{Q}}}$ ($M^{\perp_R} \supseteq M^{\perp_R}$);

(ii) $I \subseteq I^{\perp_{\mathbf{Q}} \perp_R}$ and $M \subseteq M^{\perp_R \perp_{\mathbf{Q}}}$;

(iii) $I^{\perp_{\mathbf{Q}}} = I^{\perp_{\mathbf{Q}} \perp_R \perp_{\mathbf{Q}}}$ and $M^{\perp_R} = M^{\perp_R \perp_{\mathbf{Q}} \perp_R}$;

(iv) for an arbitrary family $\{M_\alpha\}$ of subsets of $\mathbf{Q}$ and $\{I_\alpha\}$

of R, $(\bigcup_\alpha M_\alpha)^{\perp R} = \bigcap_\alpha M_\alpha^{\perp R}$ and $(\bigcup_\alpha I_\alpha)^{\perp Q} = \bigcap_\alpha I_\alpha^{\perp Q}$.

<u>Proof</u>: These properties are easily derived from the definitions. We prove the latter parts of (ii), (iii) and (iv) as illustrations. For (ii), consider $m \varepsilon M$ and $r \varepsilon M^{\perp R}$, then $r*m = 0$. This entails that m annihilates r and hence $m \varepsilon M^{\perp R \perp Q}$. Let $I = M^{\perp R}$, then $M^{\perp R} \subseteq M^{\perp R \perp Q \perp R}$ by (ii). Applying (i), we also have $M^{\perp R} \supseteq M^{\perp R \perp Q \perp R}$ and so (iii) also follows. Now let $a \varepsilon \bigcap_\alpha M_\alpha^{\perp R}$. This means that $a \varepsilon M_\alpha^{\perp R}$ for all α. If $b \varepsilon M_\alpha$, $a*b = 0$ so that $a*b = 0$ for all $b \varepsilon \bigcup_\alpha M_\alpha$. Thus, $a \varepsilon (\bigcup_\alpha M_\alpha)^{\perp R}$. Suppose $a \varepsilon (\bigcup_\alpha M_\alpha)^{\perp R}$ and $b \varepsilon M_\alpha$. Then $b \varepsilon \bigcup_\alpha M_\alpha$ and so $a*b = 0$. Hence $a \varepsilon M_\alpha^{\perp R}$ for all α and we have $(\bigcup_\alpha M_\alpha)^{\perp R} \subseteq \bigcap_\alpha M_\alpha^{\perp R}$. Q.E.D.

Lemma 3.1 is of particular interest as an algebraic result as well as being the starting point for investigation of a particular type of modules. For example, if R is an associative ring with identity and Q the dual space of an R-module, lemma 3.1 corresponds to relations necessary to begin an examination of so-called "perfect dual" modules as studied by Dieudonné [29]. We shall return to this in §5, but now we provide a simple example to show that equality need not be obtained in (ii) of lemma 3.1.

<u>Example</u>: Let $A = L^1(T)$ and $B = L^\infty(T)$; that B is an algebraic A-module with respect to convolution is well-known (in fact, it is a Banach $L^1(T)$-module). Consider $C(T)$ as a submodule of $L^\infty(T)$, then it is clear that $C(T)^{\perp A \perp B} = L^\infty(T)$ because $C(T)^{\perp A} = \{0\}$. Hence, $C(T) \subseteq C(T)^{\perp A \perp B}$.

Now to our immediate objective of defining the spectrum.

Of the various ways in which the concept of spectrum has been defined (see, for example, Beurling [10], Godement [48], and the survey article by Herz [58]) the definition most readily generalized is the one considered by Godement for $L^\infty(G)$ with the weak-star topology (regarded as the dual of $L^1(G)$). The following definition is due to Domar [30, p. 3]. It is of interest to note that Edwards [38], Forelli [67] and Muhly [84] utilize essentially the same definition without reference to [30].

Definition 3.2. Let B be a Banach A-module and $b \in B$. The spectrum of b is the set

$$\{\hat{x} \in \Delta(A) : \hat{a}(\hat{x}) = 0 \text{ for all } a \in b^{\perp_A}\}.$$

For a submodule M of B, the spectrum of M is the set

$$\{\hat{x} \in \Delta(A) : \hat{a}(\hat{x}) = 0 \text{ for all } a \in M^{\perp_A}\}.$$

We denote these by $sp(b)$ and $sp(M)$ respectively.

Remarks 3.2

1. For $b \in B$ and M a submodule of B, we have $sp(b) = \text{hull}(b^{\perp_A})$ and $sp(M) = \text{hull}(M^{\perp_A})$, hence both are closed.

2. That definition 3.2 extends the definition of L^∞-spectrum as given by Godement is evident by recalling the definition of $sp(\varphi)$ for $\varphi \in L^\infty(G)$, that is $sp(\varphi) = \{\hat{x} \in \hat{G} : \hat{f}(\hat{x}) = 0 \text{ for all } f \in L^1(G)$ with $f*\varphi = 0\}$. It suffices to observe that $L^\infty(G)$ is a Banach $L^1(G)$-module with $\Delta(L^1(G)) \simeq \hat{G}$.

3. The definition of spectrum is dependent on the module operation $*_B$, and we should more appropriately write $sp(b, *_B)$. However, we

choose the former notation for convenience, and the meaning is understood within the context of the "proper" module operation. In order to exhibit this dependence, we consider as an example the C(T)-module C(T) with respect to convolution and pointwise multiplication. For $1 \varepsilon C(T)$, we have $1^{\perp} = \{f \varepsilon C(T) : f*_c 1 = 0\}$. If $*_c$ is convolution, then $1^{\perp} \not\equiv \{0\}$, yet for $*_c$ being pointwise multiplication, $1^{\perp} \equiv \{0\}$. Hence, $sp(1, \cdot) \equiv T$, but $sp(1, *) \subseteq Z$. In fact, $\Delta(C, *)$ is not the same as $\Delta(C, \cdot)$.

4. For $b \varepsilon B$, $sp(b)$ is the largest closed set where the Gelfand transforms of all the $a \varepsilon b^{\perp_A}$ vanish.

We cite Domar's work [30], in particular Chapter IV, for other definitions of spectrum in group algebra modules (see also [31], [32], [33], and Lindahl [79]). Furthermore, we point out that his investigations take a <u>different direction</u> than ours.

Evidently, the definition of spectrum is independent of the topology on B and consequently so are the properties of the spectrum. However, the particular topology used will have bearing on spectral synthesis considerations.

§3. Fundamental Properties of Spectra

The basic properties obtained in this section are fundamental in two essential ways. First, they underlie the behavior of spectra and provide a basis for a spectral synthesis theory. Secondly, these properties serve as indispensable tools for proving more profound results. The basic properties of spectra have been recognized in specialized situations and utilized by various authors from Wiener [106] and Beurling [8], [10] to more recent applications by Leaf [77], Forelli [43], and Muhly [84], [85] to mention but only a few. We re-affirm their validity in modules (compare [30]).

Our first result utilizes compatibility of the module and algebra operations and confirms our intuition that spectrum is an extension of the concept of support.

Proposition 3.2. Let B be a Banach A-module and $b \in A \cap B$. The support of $\hat{b}$ coincides with $sp(b)$.

Proof: Let $b \in A \cap B$ and $\hat{x} \notin \sigma(b)$. By the regularity of A, there is an $a \in A$ such that $\sigma(a) \cap \sigma(b) = \phi$ with $\hat{a}(\hat{x}) \neq 0$ (I, §3, Fact 3). Thus, we have that $\sigma(ab) \subseteq \sigma(a) \cap \sigma(b) = \phi$. But this means that $\widehat{ab} \equiv 0$. By semisimplicity and compatibility $a*b = 0$ and so $a \in b^{\perp_A}$. Since $\hat{a}(\hat{x}) \neq 0$, $\hat{x} \notin sp(b)$ and hence, $sp(b) \subseteq \sigma(b)$.

For the reverse inclusion, suppose $a \in b^{\perp_A}$. Let $\hat{x} \in \sigma(b)$. Now there is a net $\langle \hat{x}_\alpha \rangle \subset \Delta(A)$ such that $\hat{b}(\hat{x}_\alpha) \neq 0$ for all α and $\hat{x}_\alpha \to \hat{x}$ since $\sigma(b) = c\ell\{\hat{y} \in \Delta(A) : \hat{b}(\hat{y}) \neq 0\}$. But $0 = a*b = ab$ entails $\hat{a}\hat{b} = 0$. Since $\hat{b}(\hat{x}_\alpha) \neq 0$ for all α, $\hat{a}(\hat{x}_\alpha) = 0$ for all α and so $\hat{x}_\alpha \in sp(b)$ for all α. The facts that $sp(b)$ is closed and $\hat{x}_\alpha \to \hat{x}$ imply $\hat{x} \in sp(b)$. Q.E.D.

We now observe that spectra obey a <u>permanence property</u>. For the case of Beurling algebras see Reiter [92, 7]. Let A_1 and A_2 be Banach algebras which satisfy A1.

Proposition 3.3. Let B_2 be a Banach A-module and B_1 a Banach A_1-module with respect to the same operation and satisfying $B_2 \supseteq B_1$, $A_2 \subseteq A_1$. The spectrum of $b \in B_1$ is equal to the spectrum of b as an element of B_2.

Proof: We first observe that $b^{\perp A_2} = b^{\perp A_1} \cap A_2$ since the module operations coincide. By the inclusion reversing property of the hull operation we have $sp_2(b) \equiv \text{hull}(b^{\perp A_2}) \supseteq \text{hull}(b^{\perp A_1}) \equiv sp_1(b)$. For the opposite inclusion, suppose $\hat{x} \notin sp_1(b)$. As in the proof of proposition 3.1, the regularity of A_2 entails the existence of an $a \in A_2$ such that $\hat{a}(\hat{x}) \neq 0$ and $\sigma(a)$ is compact with $\sigma(a) \cap sp(b) = \phi$. But also $a \in A_1$, and so $a *_{B_1} b = a *_{B_2} b = 0$ with $\hat{a}(\hat{x}) \neq 0$. Hence, $\hat{x} \notin sp_2(b)$ and $sp_2(b) \subseteq sp_1(b)$. Q.E.D.

Corollary 3.4. Let B_1 and B_2 be Banach A-modules with respect to the same module operation such that $B_1 \cap B_2$ is a Banach A-module. If $b \in B_1 \cap B_2$, then $sp_1(b) = sp_2(b)$ (using notation as in the proof of 3.3).

Proof: Apply proposition 3.3 to the Banach A-module $B_1 \cap B_2 \subseteq B_1$ to get $sp(b) = sp_1(b)$ where $sp(b)$ is the spectrum of b as an element in $B_1 \cap B_2$. Repeating for $B_1 \cap B_2 \subseteq B_2$, we obtain $sp(b) = sp_2(b)$. Q.E.D.

Remarks 3.4

1. Observing that for $b \in B = A$, A regarded as a Banach A-module, $sp(b) = \sigma(b)$, and applying corollary 3.4 we obtain an alternate proof of proposition 3.2.

2. The case $B_1 \subseteq B_2$ and $A_1 = A_2$ in proposition 3.3 shows that spectra are invariant with respect to the module providing the module operations coincide. Thus, one is allowed to regard elements of B_1 as elements of B_2 without affecting the spectrum of the element.

We now consider an Abstract Wiener Tauberian Theorem. See [30, p.

32] for a formulation in the group algebra module context.

Theorem 3.5. Let B be a Banach A-module. If $b \in B$, then $sp(b) = \phi$ if and only if $b \equiv 0$.

Proof: For the necssity part of the conclusion, suppose $b \equiv 0$. Now $b^{\perp_A} = A$ and by the properties of $\hat{A}$, if $\hat{x} \in \Delta(A)$, there exists an $a \in A$ such that $\hat{a}(\hat{x}) \neq 0$ and $\hat{a}(\hat{y}) = 0$ for a fixed $\hat{y} \neq \hat{x}$. But then $\hat{x} \notin hull(A)$. Since $\hat{x}$ is arbitrary, $sp(b) = hull(A) = \phi$.

On the other hand, suppose $sp(b)$ is empty, then $hull(b^{\perp_A}) = \phi$. Let a_o be any element in A_c with $\sigma(a_o) = F$. Since $hull(b^{\perp_A}) \cap F = \phi$, I, §3, Fact 3 applies to give an element $a \in b^{\perp_A}$ such that $\hat{a} \equiv 1$ on F and having compact support so that $aa_o = a_o$. We thus have $a_o \in b^{\perp_A}$. Since a_o is an arbitrary element of A_c, $A_c \subseteq b^{\perp_A}$. But then $A = \overline{A_c} \subseteq b^{\perp_A}$ by A3 entails $A = b^{\perp_A}$. Applying A6 we obtain $b \in b^{\perp_A \perp_B} = A^{\perp_B} = \{0_B\}$. Q.E.D.

Remark 3.5

Theorem 3.5 is the analogue of Wiener's Theorem (I, §3, Fact 4) and provides a firm basis for our theory.

The following (i-iii) is due to Beurling in the classical framework [30]. Proposition 3.6 (iv) was initially shown by Godement for $B = L^\infty(G)$, $A = L^1(G)$ [48]. We re-affirm these properties for Banach modules--for instance see [30, p. 32].

Proposition 3.6. Let B be a Banach A-module with $a \in A$, $b, b' \in B$ and $\alpha \in \mathbb{C}$, then

(i) $sp(a*b) \subseteq \sigma(a) \cap sp(b)$;

(ii) $sp(\alpha b) = sp(b)$

(iii) $sp(b + b') \subseteq sp(b) \cup sp(b')$;

(iv) $sp(b + b') = sp(b) \cup sp(b')$ if $sp(b) \cap sp(b') = \phi$.

Proof: (i) Suppose $\hat{x} \varepsilon sp(a*b)$. For $a_o \varepsilon b^{\perp_A}$, $a_o \varepsilon (a*b)^{\perp_A}$ and so $\hat{a}_o(\hat{x}) = 0$. Thus, $\hat{x} \varepsilon sp(b)$. If $\hat{x} \notin \sigma(a)$, then there is an $a_1 \varepsilon A$ such that $\hat{a}_1(\hat{x}) \neq 0$ and $\sigma(a_1) \cap \sigma(a) = \phi$. Therefore, $aa_1 = 0$ since $\sigma(aa_1) = \phi$ and A is semisimple. Now $a_1*(a*b) = (a_1a)*b = 0$ and hence $\hat{a}_1(\hat{x}) \neq 0$ implies $\hat{x} \notin sp(a*b)$. We, therefore, obtain $sp(a*b) \subseteq \sigma(a)$.

(ii) This is immediate since $a*(\alpha b) = 0$ if and only if $a*b = 0$.

(iii) Let $\hat{x} \notin sp(b) \cup sp(b')$. Then there are $a_1, a_2 \varepsilon A$ with $a_1*b = 0$ and $a_2*b' = 0$ satisfying $\hat{a}_j(\hat{x}) \neq 0, j = 1, 2$. Set $a = a_1a_2$. We have $a*(b + b') = a_1a_2*(b + b') = a_2*(a_1*b) + a_1*(a_2*b') = 0$. But $\hat{a}(\hat{x}) = \widehat{a_1a_2}(\hat{x}) = \hat{a}_1(\hat{x})\hat{a}_2(\hat{x}) \neq 0$ so that $\hat{x} \notin sp(b + b')$. Hence, $sp(b + b') \subseteq sp(b) \cup sp(b')$.

(iv) Suppose $a \varepsilon (b + b')^{\perp_A}$, then $a*b = -a*b'$. Thus, applying (ii), $sp(a*b) = sp(-a*b') = sp(a*b')$. By (i), $sp(a*b) \subseteq sp(b)$ and $sp(a*b) \subseteq sp(b')$. Therefore, $sp(a*b) = \phi$ and Theorem 3.5 entails $a*b = a*b' = 0$. Since $\hat{x} \varepsilon sp(b) \cup sp(b')$ implies $\hat{a}(\hat{x}) = 0$ if $a*(b + b') = 0$, we arrive at $\hat{x} \varepsilon sp(b + b')$. The inclusion $sp(b) \cup sp(b') \subseteq sp(b + b')$ is obtained. In conjunction with (iii) this implies (iv). Q.E.D.

In case that for $\hat{x} \varepsilon \sigma(a) \cap sp(b)$ there exists a net $\langle \hat{x}_\alpha \rangle$ in $[cosp(a)]^c \cap sp(b)$ such that $\hat{x}_\alpha \to \hat{x}$, then equality obtains in (i). For example, this is true if $b \varepsilon A \cap \mathcal{B}$.

The significance of the existence of local units in the Banach algebra A is evident in the proofs of the preceding properties. It is not incidental that Banach modules also enjoy this property.

Proposition 3.7 (Local Units). Let B be a Banach A-module and $b \in B$.

(i) If $a \in A$ is such that $\hat{a} \equiv 0$ on a nbhd. of $sp(b)$, then $a*b = 0$.

(ii) If $a \in A$ is such that $\hat{a} \equiv 1$ on a nbhd. of $sp(b)$, then $a*b = b$.

Proof: (i) By proposition 3.6 (i), $sp(a*b) \subseteq \sigma(a) \cap sp(b) = \phi$. Applying Theorem 3.5, $a*b = 0$.

(ii) Suppose $a_1 \in A$ satisfies $\hat{a}_1 \equiv 1$ on W, a nbhd. of $sp(b)$. Let $a_o \in A$. Now set $a = a_1a_o - a_o$. This entails $\hat{a} \equiv o$ on W. By (i), $a*b = 0$. Therefore,

$$a_o*(a_1*b - b) = a_o*(a_1*b) - a_o*b = (a_oa_1 - a_o)*b = a*b = 0.$$

We then have $a_1*b - b \in A^{\perp B}$ since $a_o \in A$ is arbitrary. By A5 $a_1*b - b = 0$. Q.E.D.

It is evident that if $b \in B$ has compact spectrum, then $b \in B_e$ (recall that B_e is the essential part of B). That is, if $sp(b)$ is compact, we can find an $a \in A$ satisfying $\hat{a} \equiv 1$ on a relatively compact nbhd. of $sp(b)$ and proposition 3.7 (ii) entails $a*b = b$. Thus, $b \in A*B \subseteq B_e$.

Indeed, proposition 3.7 is one of the most frequently applied properties of spectra. In fact, we obtain a partial converse to proposition 3.6 (iii) as an application. We remark that the proof of the existence part of proposition 3.8 (below) is due to R. E. Edwards [38]. An alternate proof of uniqueness is also available there.

Proposition 3.8. Let B be a Banach A-module and $b \in B$. Suppose E_1 and E_2 are disjoint compact sets in $\Delta(A)$ with $sp(b) \subseteq E_1 \cup E_2$, then there exists a unique decomposition $b = b_1 + b_2$ where

$sp(b_i) \subseteq E_i$, $i = 1, 2$.

Proof: Let U_1 and U_2 be disjoint relative compact nbhds. of E_1 and E_2 respectively. Let $a_i \in A$ be such that $\hat{a}_i \equiv 1$ on W_i, a nbhd. of E_i satisfying $E_i \subset W_i \subset \sigma(a_i) \subset U_i$, $i = 1, 2$. Set $b_i = a_i * b$. Then $\widehat{a_1 + a_2} = \hat{a}_1 + \hat{a}_2 = 1$ on $W_1 \cup W_2$ with $sp(b_1 + b_2) \subseteq sp(b_1) \cup sp(b_2) \subseteq E_1 \cup E_2 \subseteq W_1 \cup W_2$. By proposition 3.7 (ii),

$$b_1 + b_2 = (a_1 + a_2)*(b_1 + b_2) \quad \text{and} \quad b = (a_1 + a_2)*b.$$

Therefore, $b = a_1*b + a_2*b = b_1 + b_2$.

For uniqueness, suppose b has two decompositions $b = b_1 + b_2$ and $b = c_1 + c_2$ with c_i and b_i having spectra in E_i, $i = 1, 2$. Then $c_1 + c_2 = b_1 + b_2$ implies $c_1 - b_1 = b_2 - c_2$ so that $sp(c_1 - b_1) = sp(b_2 - c_2)$. But then $sp(c_i - b_i) \subseteq E_1 \cap E_2 = \phi$, $i = 1, 2$. Hence, $c_1 - b_1 = b_2 - c_2 = 0$ by Theorem 3.5. Q.E.D.

Our next result is a General Summability Theorem (for example see Helson [56, p. 66] and Forelli [43, p. 37]).

Theorem 3.9. Let B be an essential Banach A-module. The set $B_o \equiv \{b \in B : sp(b)$ is compact$\}$ is (norm-) dense in B.

Proof: Let $b \in B$. Now let $\langle e_\alpha \rangle \subset A$ be an approximate identity for A. Since B is essential, there is a $b_1 \in B$ and an $a_1 \in A$ with $b = a_1*b_1$. Thus,

$$\begin{aligned} ||e_\alpha*b - b||_B &= ||e_\alpha*(a_1*b_1) - a_1*b_1||_B \\ &= ||(e_\alpha a_1)*b_1 - a_1*b||_B \\ &\leq C||e_\alpha a_1 - a||_A||b||_B \to 0. \end{aligned}$$

By A3, we can choose $\langle e_\alpha \rangle \subset A_c$, i.e., an approximate identity with compactly supported Gelfand transforms. By proposition 3.7 (i), $sp(e_\alpha * b) \subseteq \sigma(e_\alpha)$ and so $sp(e_\alpha * b)$ is compact. Hence, $e_\alpha * b \in B_o$. The fact that $||e_\alpha * b - b||_B \to 0$ implies B_o is dense in B. Q.E.D.

Remark 3.9

For $A = L^1(G)$ and $B = C(G)$, G a compact abelian group, Theorem 3.9 shows that the trigonometric polynomials are dense in C(G). Inherent in 3.9 are questions regarding trigonometric approximation and existence of approximate identities for A.

We now consider an alternate way of looking at the spectrum of a submodule. Recall that $sp(M) = hull(M^{\perp A})$ for a submodule M of B. Here, we view the spectrum as the union of the spectra of all its elements. This perspective is due to K. de Leeuw and H. Mirkil in their investigation of $C_o(G)$ [23].

Proposition 3.10. Let B be a Banach A-module and M a submodule of B, then $sp(M) = c\ell(\bigcup_{b\varepsilon M} sp(b))$.

Proof: Suppose $b \varepsilon M$. Now $b^{\perp A} \supseteq M^{\perp A}$ and so $hull(b^{\perp A}) \subseteq hull(M^{\perp A})$. Thus, $sp(b) \subseteq sp(M)$ and it follows that $\bigcup_{b\varepsilon M} sp(b) \subseteq sp(M)$. Since M is closed, $c\ell(\bigcup_{b\varepsilon M} sp(b)) \subseteq sp(M)$.

For the reverse inclusion, let $\hat{x} \notin c\ell(\bigcup_{b\varepsilon M} sp(b))$. Now there is a nbhd. U of $\hat{x}$ which is disjoint from $c\ell(\bigcup_{b\varepsilon M} sp(b))$ and hence from $\bigcup_{b\varepsilon M} sp(M)$. Let $a \varepsilon A$ be such that $\hat{a}(\hat{x}) \neq 0$ and $\sigma(a) \subset U$. By proposition 3.7 (i), we have that $sp(a*b) \subseteq \sigma(a) \cap sp(b) = \phi$ for every

$b \in M$. Hence, $a*b = 0$ for all $b \in M$ or equivalently $a \in M^{\perp_A}$. Since $\hat{a}(\hat{x}) \neq 0$, $\hat{x} \notin sp(M)$. We therefore get $sp(M) \subseteq c\ell(\bigcup_{b\in M} sp(M))$.

Q.E.D.

We are able to obtain $\bigcup_{b\in M} sp(b)$ closed under additional hypothesis.

Lemma 3.11. Let B be a Banach A-module and M any submodule of B. If $\hat{x} \in \bigcup_{b\in M} sp(b)$, then for any neighborhood U of $\hat{x}$, there is a nonzero $b \in M$ with $sp(b) \subset U$.

Proof: Let $x \in \bigcup_{b\in M} sp(b)$. Now there is a $b_o \in M$ such that $\hat{x} \in sp(b_o)$. Let U be a nbhd. of $\hat{x}$ and $a \in A$ with $\hat{a}(\hat{x}) \neq 0$, $\sigma(a) \subset U$. Set $b = a*b_o$. Clearly $b \in M$ and $b \neq 0$ with $sp(b) \subseteq \sigma(a) \subset U$.

Q.E.D.

Proposition 3.12. Let B be a Banach A-module and M a closed submodule of B. If $\Delta(A)$ is metrizable, then $sp(M) = \bigcup_{b\in M} sp(b)$.

Proof: By proposition 3.10, it suffices to show $\bigcup_{b\in M} sp(b)$ is closed. Suppose $\hat{x} \in c\ell(\bigcup_{b\in M} sp(b))$. Choose $\langle \hat{x}_n \rangle \subset \bigcup_{b\in M} sp(b)$ such that $\hat{x}_n \to \hat{x}$. Let V_n be a nbhd. of $\hat{x}_n$, $n = 1, 2, \cdots$, satisfying (1) the $\overline{V}_n$ are pairwise disjoint, and (2) any nbhd. U of $\hat{x}$ eventually contains the $\overline{V}_n$. By lemma 3.11, there is a $b_n \in M$, $b_n \neq 0$ with $sp(b_n) \subset V_n$ for each $n = 1, 2, \cdots$. We now have that for any nbhd. U of $\hat{x}$, $sp(\sum_{m+1}^{N} b_n) \subset \bigcup_{m+1}^{N} \overline{V}_n \subset U$. By (1), we obtain that $\langle \sum_{n=1}^{N} b_n ; N \in Z \rangle$ is Cauchy and hence $\Sigma b_n \to b$ for some $b \in B$. But M is closed and therefore $b \in M$. To complete the proof, we need to show $\hat{x} \in sp(b)$.

Let W be a nbhd. of $\hat{x}$. Now there is an n such that $\overline{V}_n \subset W$.

But $sp(b_n) \subset V_n$ and so there is an $a \in A$ with $\sigma(a) \subset V_n$ and $a*b = 0$. Choose a such that $\sigma(a) \supseteq sp(b_n)$. Thus, we have $a*b = a*(\sum_n b_n) = \sum_n (a*b_n) = a*b_n$, since $\sigma(a) \subset V_n$, $sp(b_m) \subset V_m$, $V_n \cap V_m = \phi$ if $n \neq m$, and "$*$" is continuous. Now $a \in b^{\perp A}$ implies $a \in b_n^{\perp A}$ for all n. Hence, $hull(b^{\perp A}) \supseteq hull(b_n^{\perp A})$ for all n, and $x_n \in sp(b_n) = hull(b_n^{\perp A})$ for all n. This with $\hat{x}_n \to \hat{x}$ and the hull closed gives $\hat{x} \in sp(b)$. Q.E.D.

We now view the spectrum of elements in terms of the submodules generated by them. This will play an interesting role in consideration of spectral synthesis (III, §5, IV and V).

<u>Proposition 3.13</u>. Let B be a Banach A-module. If $b \in B$, then

$sp(b) = sp[b]$.

<u>Proof</u>: We first observe that $b^{\perp A} \subseteq (A*b)^{\perp A}$. Suppose $a \in (A*b)^{\perp A}$. For any $a_1 \in A$, $a*(a_1*b) = 0$ and thus, $a_1*(a*b) = 0$. Since a_1 is arbitrary, $a*b \in A^{\perp B}$. Hence, A4 implies $a*b = 0$, so that $a \in b^{\perp A}$. We therefore have $b^{\perp A} = (A*b)^{\perp A}$ which entails $sp(b) = hull(b^{\perp A}) = hull((A*b)^{\perp A}) = hull([b]^{\perp A}) = sp([b])$ since $(A*b)^{\perp A} = [b]^{\perp A}$. Q.E.D.

<u>Remarks 3.13</u>

1. It is not difficult to see that we also have $\overline{[b]}^{*\perp A} = [b]^{\perp A}$, and hence, $sp(b) = sp(\overline{[b]}^{*})$ as well.

2. The proof of the relation $b^{\perp A} = [b]^{\perp A}$ exhibits the recurrent technique in application of A4.

3. In the classical modules, say $B = L^p(T)$, $A = L^1(T)$, $1 \leq p \leq \infty$, care must be taken in consideration of the smallest closed translation

invariant subspace containing the element f as being the closed submodule generated by f. Recall that the concepts are the same for $1 \leq p < \infty$ and the norm topology (see I, §4, Fact 9 for L^1), yet it is not true that they coincide for the case $p = \infty$ unless the weak-star topology is used. This is another factor in the choice of a weak topology for general modules (see I, §6).

4. We emphasize that $[b] = \overline{A*b}$ and $\overline{[b]}^* = \overline{A*b}^*$. This is <u>not</u> the algebraic definition of "submodule generated by b." In fact, one cannot use the algebraic definition

$$(b) = \{a*b + nb : a \in A, n \in Z\}$$

because it need not coincide with $[b]$. It is known that if $\Delta(A)$ is connected, A contains no <u>algebraically</u> finitely-generated ideals with an approximate identity other than $\{0\}$ and itself. But this is certainly not true for our definition of "generated by an element." We refer to Dietrich [27], [28] and Altzmon [1], [3] for results along this line.

Our next result characterizes annihilator submodules of a special type of deal in terms of spectra. This will be important for our future invesitgation of spectral synthesis in modules. Before stating our result, let us recall that for a closed set E of $\Delta(A)$, $J(E) = c\ell\{a \in A : \hat{a} \equiv 0$ on a nbhd. of $E\}$.

<u>Theorem 3.14</u>. Let B be a Banach A-module, E a closed subset of $\Delta(A)$, and $b \in B$. Then $b \in J(E)^{\perp B}$ if and only if $sp(b) \subseteq E$.

<u>Proof</u>: Suppose $b \in J(E)^{\perp B}$ and $\hat{x} \in sp(b)$. Now if we take any $a \in J(E)$, then $a*b = 0$, and so $\hat{a}(\hat{x}) = 0$. Since this entails that

$\hat{a}(\hat{x}) = 0$ for all $a \in J(E)$, $\hat{x} \in \mathrm{hull}(J(E)) = E$.

Suppose now that $sp(b) \subseteq E$ and $a \in A$ satisfies $\hat{a} \equiv 0$ on a nbhd. of E. By proposition 3.7 (i), $a*b = 0$. But $\{a \in A : \hat{a} \equiv 0$ on a nbhd. of $E\}$ is dense in $J(E)$ so that $a*b = 0$ for all $a \in J(E)$. Thus $b \in J(E)^{\perp B}$. Q.E.D.

Observe that the conclusion of the theorem is equivalent to

$$J(E)^{\perp B} = \{b \in B : sp(b) \subseteq E\}.$$

Corollary 3.16. Let B be a Banach A-module. Suppose $\{e_\alpha\}$ is a family of closed subsets of $\Delta(A)$, then $J(\bigcap_\alpha E_\alpha)^{\perp B} = \bigcap_\alpha J(E_\alpha)^{\perp B}$.

Proof: If $b \in J(\bigcap_\alpha E_\alpha)^{\perp B}$, then $sp(b) \subseteq \bigcap_\alpha E_\alpha$ by theorem 3.15 and so $sp(b) \subseteq E_\alpha$ for all α. Consequently, $b \in J(E_\alpha)^{\perp B}$ for all α. The reverse conclusion follows by reversing the argument. Q.E.D.

Remark 3.16

The properties of spectra have required no "particular" topology on B, and indeed, these fundamental properties are independent of the topology on B as remarked earlier (§2). With the exception of Theorem 3.9 and proposition 3.13 no mention of the topology on B is required.

We have, therefore, observed that our concept of spectrum satisfies the properties that are fundamental for a spectral synthesis theory analogous to that of the weak-star spectral synthesis of bounded functions (see Chapter II) as initiated by Beurling [8]. Furthermore, this basic behavior is the same as recognized by Domar [30], Herz [59], et al.

We conclude this section with some properties of spectral particular to a class of group algebra modules. For the remainder of this section,

<u>we assume $A = A(G)$ is a Banach convolution subalgebra of $L^1(G)$, and $B = B(G)$ is a Banach space of functions (measures, distributions) on G which is a Banach A-module with respect to convolution</u>.

We write "B has involution" if B has an involution $\varphi \to \tilde{\varphi}$ agreeing with the $L^1(G)$ involution, i.e., $\tilde{\varphi}(x) = \overline{\varphi(-x)}$.

Although examples are to be given shortly (II, §4), we mention that one can take the following objects for A and B.

1. $A = L^1(G)$ and $B = H(G)$, $H(G)$ a homogeneous Banach space, hence in particular, $H(G)$ can be any Segal algebra. We refer to Katznelson [67], Wang [103], [104], Bennett and Gilbert [6], Reiter [92], [93] and Hewitt and Ross [61] for discussions of these type of spaces.

2. $A = A(G)$ a Beurling algebra and B any normed ideal in the sense of Cigler [20] (see II, §4).

3. $A = L^1(G)$, $B = \overline{L(G)}$ as defined by Wermer [105]. See Burnham [17] and III, §4 (6).

<u>Convention</u>

We identify $\hat{x} \in \Delta(A)$ with the "associated function." For instance, if $\hat{x} \in \hat{G}$, then by $\hat{x} \in L^\infty(G)$ is meant the map $\hat{x} \to (x,\hat{x})$.

We shall refer to these modules as <u>algebra modules</u>. We proceed to the <u>algebraic properties of spectra</u> for these modules.

<u>Proposition 3.17</u>. Let B be an algebra A-module. The following properties are satisfied:

(i) $sp(L_x\varphi) = sp(\varphi)$ if $x \in G$ and $\varphi \in B$;

(ii) $sp(\hat{x}\varphi) = \hat{x} + sp(\varphi)$ if $\hat{x}\varphi \in B$ for $\hat{x} \in G$, $\varphi \in B$ and $\hat{x}A \subseteq A$.

(iii) $sp(\bar{\varphi}) = -sp(\varphi)$ if $\varphi \varepsilon B$ and $\bar{\varphi} \varepsilon B$, and $f*\bar{\varphi} = (\bar{f}*\varphi)^-$ for all $f \varepsilon A$;

(iv) $sp(\tilde{\varphi}) = sp(\varphi)$ if $\varphi \varepsilon B$, $\tilde{\varphi} \varepsilon B$ where B has involution and $f*\tilde{\varphi} = (f^* * \varphi)^-$ for all $f \varepsilon A$;

(v) $(\varphi \cdot \psi) \subseteq c\ell[sp(\varphi) + \sigma(\psi)]$ for $\varphi \varepsilon B$, $\psi \varepsilon A$ and $\varphi \cdot \psi \varepsilon A$, $\hat{x}\psi \varepsilon A$ for all $\hat{x} \varepsilon G$ where B has involution;

(vi) $sp(\varphi \cdot \psi) \subseteq c\ell[sp(\varphi) + sp(\psi)]$ if $\varphi, \psi \varepsilon B$ and $\varphi \cdot \psi \varepsilon B$ where B has involution.

Proof: (i) for $f \varepsilon A$ and $\varphi \varepsilon B$, $f*L_x\varphi = L_x(f*\varphi)$. Thus, $(L_x\varphi)^{\perp A} = \varphi^{\perp A}$. This entails $sp(L_x\varphi) = sp(\varphi)$.

(ii) Let $\hat{y} \varepsilon sp(\hat{x}\varphi)$ and $f*\varphi = \varphi$. Now $\bar{\hat{x}}f*\hat{x}\varphi = \varphi$ and hence $(\bar{\hat{x}}f)^\wedge(\hat{y}) = 0$. But then $(L_x\hat{f})(\hat{y}) = 0$. Therefore, $\hat{f}(\hat{y} - \hat{x}) = 0$ and $\hat{y} - \hat{x} \varepsilon sp(\varphi)$. We obtain $sp(\varphi) + \hat{x} \subseteq sp(\hat{x}\varphi)$. For the reverse inclusion, let $\hat{y} \varepsilon sp(\varphi)$ and $f*\hat{x}\varphi = 0$. Now $\hat{x}f*\varphi = 0$ and so $(\hat{x}f)^\wedge(\hat{y}) = 0$. This implies that $L_{\hat{x}}\hat{f}(\hat{y}) = 0$ and thus $\hat{f}(\hat{y} - \hat{x}) = 0$. Now $\hat{y} - \hat{x} \varepsilon sp(x\varphi)$ entails $sp(\varphi) \subseteq \hat{x} + sp(\hat{x}\varphi)$ and the proof of (ii) is complete.

(iii) We have $f*\bar{\varphi} = 0$ if and only if $\bar{f}*\varphi = 0$ by hypothesis. By direct calculation, $\hat{\bar{f}}(\hat{x}) = \hat{f}(-\hat{x})$. Therefore, $\hat{x} \varepsilon sp(\varphi)$ implies that if $f*\varphi = 0$, then $\overline{f*\varphi} = 0$ and $\hat{\bar{f}}(\hat{x}) = 0$. Hence, $\hat{f}(-\hat{x}) = 0$ and $-x \varepsilon sp(\varphi)$. If $\hat{x} \varepsilon sp(\varphi)$, then for $f*\bar{\varphi} = 0$, $\bar{f}*\varphi = 0$, $\bar{f}*\varphi = 0$ so that $0 = \hat{\bar{f}}(\hat{x}) = \hat{f}(-\hat{x})$. This means $-\hat{x} \varepsilon sp(\bar{\varphi})$. Therefore, $-sp(\varphi) = sp(\bar{\varphi})$.

(iv) By hypothesis, $f*\tilde{\varphi} = 0$ if and only if $f^* * \varphi = 0$. Therefore, $\hat{x} \varepsilon sp(\tilde{\varphi})$ implies for $f*\varphi = 0$ that $f^* * \tilde{\varphi} = 0$ and so $f^*(\hat{x}) = 0$. But $f^*(\hat{x}) = \bar{\hat{f}}(\hat{x})$ entails $\hat{f}(\hat{x}) = 0$, i.e. $\hat{x} \varepsilon sp(\varphi)$. If $\hat{x} \varepsilon sp(\varphi)$, then for $f*\tilde{\varphi} = 0$, we have $f^* * \varphi = 0$ and hence $f^*(\hat{x}) = 0$ implies $\overline{\hat{f}}(\hat{x}) = 0$. Consequently, $\hat{f}(\hat{x}) = 0$ and $\hat{x} \in sp(\tilde{\varphi})$. We obtain $sp(\varphi) = sp(\tilde{\varphi})$.

(v) We note that $\varphi\cdot\psi \in A \cap B$ so that $\sigma(\varphi\psi) = sp(\varphi\psi)$. Suppose $(\hat{x} + \sigma(\psi)) \cap (-sp(\varphi)) = \emptyset$, then (iii) entails

$$sp[(\psi\hat{x})^* * \overline{\varphi}] \subseteq \sigma((\psi\hat{x})^*) \cap sp(\overline{\varphi}) = \sigma((\psi\hat{x})^*) \cap (-sp(\varphi)) = \emptyset.$$

Therefore, $(\psi\hat{x})^* * \overline{\varphi} = 0$. If $\hat{x} \notin [sp(\psi) + \sigma(\psi)]$, we obtain $(\varphi\cdot\psi)^\wedge(\hat{x}) = 0$. (Recall $(\hat{x} + \sigma(\psi)) \cap (-sp(\varphi)) = \emptyset$) since $(\psi\hat{x})^* * \varphi = 0$ implies $\psi\varphi*\hat{x}(0) = 0$. We conclude that $cosp(\psi\hat{x})^c \subseteq [sp(\varphi) + \sigma(\psi)]$, hence $\sigma(\psi\hat{x}) \subseteq c\ell[sp(\varphi) + \sigma(\psi)]$.

(vi) Suppose $\psi \in B$ and $\varphi \in B$. Let U be a relatively compact nbhd. of zero. Choose $f \in A$ such that $\hat{f}(0) \neq 0$ and $\sigma(f) \subset U$. By the fact that $\hat{x} \notin -sp(\varphi) + U$ implies $(\overline{\varphi}L_y f)^\wedge(\hat{x}) = 0$ for all $y \in G$, we obtain $(\overline{\varphi}\cdot L_y\cdot f\cdot\hat{y})*\psi = 0$ if $[\hat{y} - sp(\varphi) + U] \cap sp(\psi) = \emptyset$. Therefore, $(f\hat{y})^* * (\varphi\psi) = 0$ for $\hat{y} \notin - U + sp(\varphi) + sp(\psi)$ and $sp(\varphi\cdot\psi) \subset sp(\varphi) + sp(\psi) - U$. But U can be made arbitrarily small so that the conclusion follows. Q.E.D.

<u>Remarks 3.17</u>

1. Property (i) does not require B to be a space of "functions" on G, but does use the properties of the (representation of G) translation operators $\{L_x : x \in G\}$.

2. The above properties hold for the module $A = L^1(G)$, $B = L^\infty(G)$. Moreover, some of the requirements say for $A = L^1_w(G)$, $B = L^\infty_w(G)$ are non-trivial, for example, "Shilov's condition" is necessary for (v), which is encompassed by our assumption A6.

§4. Examples

We provide a variety of examples of Banach modules. Of considerable interest are the group-algebra modules, but we also deal with other types of modules. We do not prove that these are Banach modules, but refer the reader to works where verifications (if non-trivial) may be found. Evidently, one cannot expect to exhaust all possible examples, but we do hope to provide an acceptable assortment.

Example 1. Non-algebra $L^1(G)$-modules

a. Let $1 < p \leq \infty$ and G be non-compact. The spaces $(L^p(G), ||\cdot||_p)$ are Banach $L^1(G)$-modules with respect to convolution. It is well known that these spaces are not convolution algebras. For $p \neq \infty$, $L^p(G)$ is an essential $L^1(G)$-module, while for $p = \infty$ we have $L^1(G)*L^\infty(G) = C_u(G)$.

b. Let G be non-compact, then $(C_o(G), ||\cdot||_\infty)$ is a Banach $L^1(G)$-module with respect to convolution.

c. Let X be a locally compact Hausdorf space and $T : G \to X$ a strongly continuous representation of G. For $f \in L^1(G)$ and $x \in X$, define the convolution of f with x by

$$f*x = \int_G T_g x f(g) dm(g).$$

The space X is then an $L^1(G)$-module with respect to convolution and is generally not a Banach space of functions on G. If X is a Banach space, then it is a Banach $L^1(G)$-module. (X,G) is an example of a flow. For utilization of spectral properties in flows see Forelli [43

[44] and Muhly [84], [85].

We now define some important concepts as a prelude to our next class of modules. A Banach space of complex-valued measurable functions on G, $(B(G), ||\cdot||_B)$, is a homogeneous Banach space if

(i) $f \in B(G)$ and $x \in G$ imply $L_x f \in B(G)$ and $||L_x f||_B = ||f||_B$, where $L_x f(y) = f(y - x)$ for all $y \in G$;

(ii) $x \to L_x f$ is a continuous map of G onto $(B(G), ||\cdot||_B)$.

If $B(G)$ is a subalgebra of $L^1(G)$ such that $B(G)$ is a Banach algebra with respect to $||\cdot||_B \geq ||\cdot||_{L^1}$, and satisfies (i) and (ii), then $B(G)$ is a homogeneous Banach subalgebra. If $B(G)$ is a dense homogeneous Banach subalgebra, it is called a Segal algebra. We refer to Wang [104] for a discussion of homogeneous Banach spaces, and Reiter [92], [93] for Segal algebras.

Example 2. Algebra $L^1(G)$-modules

It is known that Segal algebras are Banach $L^1(G)$-modules with respect to convolution. In fact, any homogeneous Banach space is a Banach $L^1(G)$-module with respect to convolution.

a. Let G be infinite and compact. The spaces $(L^p(G), ||\cdot||_p)$, $1 \leq p < \infty$, are Segal algebras and hence convolution Banach modules of $L^1(G)$.

b. $(C(G), ||\cdot||_\infty)$ is a Segal algebra, G infinite compact.

c. Let $1 \leq k < \infty$. The spaces $C^{(k)}(T) = \{f \in C(T) : f^{(n)} \in C(T),$ $n = 0, \ldots, k\}$ with the norm $||f||_{C^{(k)}} = \sum_{n=0}^{k} \frac{1}{n!} \max_x |f^{(n)}(x)|$ are $L^1(T)$-modules.

d. $A_p(G) = \{f \in L^p(G) : \hat{f} \in L^p(G)\}$ with the norm $||f||_{A_p} = ||f||_1 + ||\hat{f}||_p$ are $L^1(G)$-modules, $1 \leq p \leq \infty$.

e. Let $1 \leq k < \infty$. Then $L^{(k)}(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : f^{(n)} \in L^1(\mathbb{R})$, $n = 0, 1, 2, \ldots, k$ and $f^{(m)}$ absolutely continuous on $\mathbb{R}$, $m = 0, 1, \ldots, k-1\}$ with the norm $||f||_{L^{(k)}} = \sum_{n=0}^{k} \frac{1}{n!} ||f^{(n)}||_1$, are $L^1(\mathbb{R})$-modules.

f. <u>The Wiener Algebra</u>: $W(\mathbb{R}) = \{f \in C(\mathbb{R}) : \sum_{n \in Z} \max_{[n,n+1]} |f(x)| < \infty\}$ is

(i) an $L^1(\mathbb{R})$-module with respect to convolution;

(ii) an $L^1(\mathbb{R}) \cap C_o(\mathbb{R})$ module with respect to both convolution and pointwise multiplication;

(iii) an $L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ module with respect to convolution $(1 \leq p \leq \infty)$;

(iv) an $A_p(\mathbb{R})$-module with respect to convolution $(1 \leq p \leq \infty)$.

See Wang [103], [104], and, in addition, Goldberg [49].

g. Let G be a non-discrete LCAG. Then $L^1 \cap C_o(G)$ and $L^1 \cap L^p(G)$ are $L^1(G)$-modules with respect to convolution.

The next examples are due to Goldberg [17].

h. Let $\langle D_N \rangle$ denote the Dirichlet kernel. Define $D(T) = \{f \in L^1(T) : \sup||D_N * f||_1 < \infty\}$ with norm $||f||_D = \sup||D_N * f||_1$; $B(T) = \{f \in L^1(T) : ||f - D_N * f||_1 \to 0\}$ with norm

$$||f||_B = \sup_N ||D_N * f||_1; \quad \text{and}$$

$S(T) = \{f \in L^1(T) : \hat{f}(n) = o\left(\frac{1}{\ln|n|}\right)$ with norm

$$||f||_S = ||f||_1 + \max_{1 \leq n \leq \infty} |\ln(n)\hat{f}(n)|$$

We have that $B(T)$ is a $D(T)$-module and that $S(T)$ is an $L^1(T)$-module (in fact, a Segal algebra) with respect to convolution. Other sources of examples can be drawn from the A-Segal algebras as defined by Burnham [14]. In fact, $B(T)$ as defined in (h) is an $D(T)$-Segal algebra. Moreover, if B is an A-Segal algebra, its <u>relative completion</u>, $\tilde{B}^A$, is a Banach A-module (Burnham [16], [17]). The relationship of $\tilde{B}^A$ and the β-topology on B is one of considerable interest (VI, §5).

<u>Example 3. Lipshitz Spaces</u>

Let $0 < \alpha < 1$ and consider the Lipshitz spaces as follows:

$$\mathrm{Lip}_\alpha(T) = \left\{ f \in C(T) : \sup_{\substack{t \\ x \neq 0}} \frac{|f(t+x) - f(t)|}{|x|^\alpha} < \infty \right\} \text{ with norm}$$

$$||f||_{\mathrm{Lip}_\alpha} = ||f||_\infty + \sup_{\substack{t \\ x \neq 0}} \frac{|f(t+x) - f(t)|}{|x|^\alpha}, \text{ \underline{and}}$$

$$\ell\mathrm{ip}_\alpha(T) = \left\{ f \in \mathrm{Lip}_\alpha(T) : \lim_{x \to 0} \sup_t \frac{|f(t+x) - f(t)|}{|x|^\alpha} = 0 \right\}$$

with norm $||f||_{\mathrm{Lip}_\alpha}$.

(i) $\mathrm{Lip}_\alpha(T)$ and $\ell\mathrm{ip}_\alpha(T)$ are Banach $L^1(T)$-modules;

(ii) $\mathrm{Lip}_\alpha(T)$ and $\ell\mathrm{ip}_\alpha(T)$ are both $C(T)$-modules with respect to convolution;

(iii) $\ell\mathrm{ip}_\alpha(T)$ is an $\mathrm{Lip}_\alpha(T)$-module with respect to convolution.

See Katznelson [67].

Example 4. Bounded Variation Module

Let $BV(T) = \{f \in C(T) : V_o^{2\pi}(f) < \infty\}$ with respect to the norm $||f||_{BV} = ||f||_{L^1} + V_o^{2\pi}(f)$. $BV(T)$ is an $L^1(T)$-module with respect to convolution. For spectral synthesis considerations in $BV(T)$ see [4].

Example 5. Beurling Algebras

Let w be a function satisfying (1) $w(x) \geq 1$ for $x \in G$; (2) $w(x + y) \leq w(x)w(y)$ for $x, y \in G$; and (3) w is measurable and bounded. Let

$$L_w^1(G) = \left\{ f \in L^1(G) : \int_G |f(x)|w(x)dx < \infty \right\}.$$

The space $L_w^1(G)$ is a Beurling algebra, and has norm $||f||_{1,w} = \int|f(x)|w(x)dw$. The dual of $L_w^1(G)$ is $L_w^\infty(G)$, the space of all complex measurable functions φ on G satisfying $||\varphi||_{\infty,w} = \operatorname*{ess\,sup}_{x \in G} \frac{|\varphi(x)|}{w(x)} < \infty$. For an extensive study of these algebras we refer to Reiter [92].

We have that

a. $L_w^\infty(G)$ is an $L_w^1(G)$-module with respect to the convolution

$$f*\varphi(y) = \int f(x)\overline{\varphi(x - y)}dx$$

b. if B is a closed ideal of a proper Beurling algebra A, then B is a Banach A-module with respect to convolution.

c. if G is non-compact and non-discrete, then for B a normed ideal of $L^1_w(G)$ is the sense of Cigler [20, p. 10], B is a Banach $L^1(G)$-module.

As a generalization of Beurling algebras, we consider the spaces defined by Wermer [105].

<u>Example 6. Wermer's Algebras</u>

Let L be a Banach space of integrable functions of G, a non-compact LCAG, and let $\overline{L}$ be defined by

$$\overline{L} \equiv \{\varphi : G \to C \mid \int_G |f(x)||\varphi(x)|dx < \infty \text{ for } f \in L\}.$$

For $f, g \in L$,

$$f*g(x) = \int_G f(x - y)g(y)dy$$

and for $f \in L$, $\varphi \in \overline{L}$

$$f*\varphi(x) = \int_G f(y)\varphi(y - x)dy.$$

Suppose L satisfies the following

(i) L contains the characteristic functions of all compact subsets of G;

(ii) if $f \in L$ and $g \in L$ with $|f(x)| = |g(x)|$ a.e., then $||f|| = ||g||$;

(iii) for a measurable f, $\int |f(x)||\varphi(x)|dx < \infty$ for all $\varphi \in \overline{L}$ implies $f \in L$;

(iv) every bounded linear functional φ on L is of the form

$$\varphi(f) = \int_G f(x)\varphi(x)dx \quad \text{for} \quad \varphi \in \overline{L},\ f \in L$$

and conversely, each $\varphi \in \overline{L}$ defines a linear functional in this manner;

(v) $T_y : f \to f_y, f_y(x) = f(x - y)$, is a bounded operator on L for each $y \in G$;

(vi) L is a Banach algebra under convolution.

We have that $\overline{L}$ is a Banach L-module and any ideal of L is an L-module. Moreover, L is an $L^1(G)$-module. For examples of particular weighted algebras see [105, p. 538].

Example 7

Let A be a commutative Banach algebra and $X = \Delta(A)$. Define multiplication for $\varphi \in C_o(X)$ and $f \in A$ by $f * \varphi = \hat{f}\varphi$, then $C_o(X)$ is a Banach A-module with respect to this operation.

We similarly can regard $L^p(\hat{G})$ as $L^1(G)$-modules by defining $f * \varphi = \hat{f}\varphi$ for $f \in L^1(G)$ and $\varphi \in L^p(\hat{G})$. Gulick, Liu and van Rooij have made an extensive study of group algebra modules including these [51], [52], [53], [54]. In [53], an operation is defined to make $L^1(X)$ and $\mathfrak{M}(X)$, X a locally compact Hausdorff space, $L^1(G)$-modules where G is not only a LCAG but a LCG of homeomorphisms on X satisfying additional conditions. It is a matter of taking an "appropriate" flow (X,G) (cf. Example 1(c)).

We remark that any A-Segal algebra is also a Banach A-module (for an extensive look at A-Segal algebras see Burnham [14], [15], [16] and references therein). Our next example (8a) is such an algebra.

Example 8. Operator Modules

a. Let A be the algebra of compact operators on a (separable)

Hilbert space. Set B = algebra of all operators of Hilbert-Schmidt type, then B is an A-module. In particular, A can be the algebra of all completely continuous operators on $L^2(\mathbb{R})$ with the usual norm. Let $B = \{T : L^2(\mathbb{R}) \to L^2(\mathbb{R}) | T$ is of H-S type$\}$, i.e., there exists a $K \in L^2(\mathbb{R} \times \mathbb{R})$ such that

$$Tf(x) = \int_{\mathbb{R}} K(x - y)f(y)dy \text{ [a.e.] for } f \in L^2(\mathbb{R}) \text{ and has norm}$$

$$||T||_B = \left\{ \int_{\mathbb{R}} \int_{\mathbb{R}} |K(x,y)|^2 dxdy \right\}^{1/2} = ||K||_{L^2(\mathbb{R} \times \mathbb{R})} .$$

b. (Kaplansky [66]) Let E be any Banach space and A a closed subalgebra of $B(E) \equiv$ bounded linear operators on E. Then E is a (left) Banach A-module with respect to $T *_B x = Tx$.

Example 9. Homogeneous Banach Algebras on T

We assume that Ω is a homogeneous Banach algebra on $D = \{z \in \mathbb{C} : |z| = 1\} = T$ (see the definition prior to example 2). In particular, we assume Ω satisfies

(i) Ω is a commutative and semisimple Banach algebra with respect to pointwise multiplication;

(ii) $\Delta(\Omega) = T$;

(iii) for every $e^{it} \in T$ and $f \in \Omega$ we have $L_t f \in \Omega$, and $e^{it} \to L_t$ is a strongly continuous representation;

(iv) $\Omega \supset C^\infty(T)$; in particular, Ω is regular.

We have that Ω is an essential $L^1(T)$-module. Particular examples include Sobolov spaces and interpolation spaces. For a detailed account and elaboration we refer to the work of Bennett and Gilbert as cited above.

§5. Problems Related to Spectra in Banach Modules

Our intention in this section is to formulate three problems in a spectral theory for Banach modules. This is to reinforce our contention that the module context is an "appropriate" setting for investigations of this nature. Thus, we present the perspective of the overall spectral synthesis and spectral analysis problems in the context of Banach modules.

Convention

We denote a topology on B, a Banach A-module, by τ. Then τ will "represent" any of the following topologies:

(i) norm topology;

(ii) strict topology (β);

(iii) $\star$-topology.

That is, the definitions and results are valid for B having the τ-topology, τ meaning any of the above topologies (i)-(iii). (In case $B = A^*$, τ may also represent the weak-star topology on B induced by A).

Spectral Synthesis

We first introduce the concepts of sets of spectral synthesis for Banach modules, and then state a "spectral synthesis problem" for such modules. Recall that the general problem of spectral synthesis in Banach algebras is difficult and seemingly intractable (see the exposition in II). In the succeeding Chapters IV and V, we exhibit the relationship to spectral synthesis in Banach algebras and obtain a Wiener-Ditkin-Shilov theorem for Banach modules.

Our first definition is a restatement of the definition of sets of spectral synthesis for Banach algebras relative to the module setting. This leads us to our formulation.

Definition 3.3 (II, §3). A closed subset E of $\Delta(A)$ is a set of spectral synthesis relative to A if there exists a unique closed ideal with hull equal to E. We write "E is an S-A set."

Definition 3.4. A closed subset E of $\Delta(A)$ is a set of spectral synthesis relative to B if there exists a unique τ-closed submodule of B with spectrum equal to E. We write "E is an S-B set."

Two cases of special interest are included in our definition 3.4.

1. If τ is the norm topology and $B = A$, then definition 3.4 coincides with definition 3.3. Therefore, our definition agrees with the usual notion of sets of spectral synthesis for Banach algebras.
2. If τ is the $*$-topology and $B = A^*$, then our definition reduces to the definition of "weak-star" spectral synthesis for duals of Banach algebras (see Katznelson [67, §7] and Domar [32]). Hence our definition of S-B sets is, indeed, a unification of the usual formulations.

Remark. Definition 3.4 applies to A-modules B more general than Banach A-modules, say locally-convex A-modules (for a discussion of such objects see Rigelhof [94]). Since the examples we deal with are Banach A-modules (cf. III, §4), we utilize the latter concept.

We are now in a position to state a spectral synthesis problem for Banach modules.

SPECTRAL SYNTHESIS PROBLEM: Given a closed subset E of $\Delta(A)$, does there exist a unique τ-closed submodule M with $sp(M) = E$?

To the author's knowledge, the spectral synthesis problem for Banach modules as introduced here has not been previously stated. There are, of course, special cases of interest in which a spectral synthesis problem has been posed. For instance, Gelfand-Raikov-Shilov [45] define such a problem for $B = A^*$ with the weak-star topology (induced by A); see case (2) after definition 3.4. Furthermore, the case $B = C_o(G)$, $A = L^1(G)$ has been considered by deLeeuw and Mirkil [23]. We point out that a spectral synthesis problem for particular group algebra modules is indicated in [61].[10]

Spectral Analysis

The second problem we formulate for Banach modules deals with the possible decomposition of τ-closed submodules into one-dimensional submodules. Kitchen [69] has considered this question for a "particular" submodule of B with the norm topology on B, _but not in terms of spectra as we suggest here_.

We formulate the problem explicity.

SPECTRAL ANALYSIS PROBLEM: Given a Banach A-module B, when is a τ-closed submodule M of B decomposable into one-dimensional submodules?

The classical case $B = C(T)$, $A = L^1(T)$ where B is given the sup norm topology is well-known, for example, see Edwards [37, §11]. The notion of almost periodicity plays a role in our context. (This is

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[10] During the preparation of this thesis, Kitchen and Robbins [70] have considered a spectral synthesis problem in Banach modules called "strongly almost periodic." We add that they do _not_ use the concept of spectrum.

Kitchen's interest in the spectral analysis of modules.) We will be able to obtain results similar to Kitchen's concerning almost periodicity but in terms of spectra, as opposed to the approach in [69].

A discussion of this problem re-enforces our contention as desired. Observe that we actually have a "problem" for each topology τ "represents." Naturally enough, alteration of topology affects any such possible decomposition. Thus, the spectral synthesis problems depend on the particular topology imposed on B. Generally, the objective is to attempt to describe the τ-closed submodules of B, and it is by determining the sets of spectral synthesis that one may but hope to progress toward a solution.

Our investigation is an attempt to unify existing theory and to render an alternate view of the spectral synthesis (analysis) problem in general. We elaborate in Chapters IV and V.

Closure Problems

We now define another problem for Banach modules which is related to spectral synthesis. The purpose again is to demonstrate that the definition of spectra in Banach modules allows spectral synthesis considerations to be regarded in a more general context.

Propositions 3.6 (iv) and 3.8 will be restated to illustrate the origin of the considerations in this section.

Proposition 3.6 (iv). Let B be a Banach A-module. If b_1 and b_2 satisfy $sp(b_1) \cap sp(b_2) = \phi$, then $sp(b_1 + b_2) = sp(b_1) \cup sp(b_2)$.

Proposition 3.8. Let B be a Banach A-module and $b \in B$. Suppose E_1 and E_2 are disjoint compact sets in $\Delta(A)$ with $sp(b) \subseteq E_1 \cup E_2$, then there exists a unique decomposition $b = b_1 + b_2$ where $sp(b_i) \subseteq E_i$, $i = 1, 2$.

Proposition 3.8 is a partial converse to 3.6 (iv). The question we concern ourselves with is to further investigate the validity of the converse of 3.6 (iv) in the context of Banach modules. The purpose of this section is to define concepts for Banach modules which permit one to do this (IV, §5). We model our study after the case $B = L^{\infty}(G)$, $A = L^1(G)$ which was studied by Reiter [91], <u>although not in terms of Banach modules</u>.

Suppose E_1 and E_2 are non-empty disjoint subsets of $\hat{G}$. If I and J are closed ideals in $L^1(G)$ with $hull(I) = E_1$ and $hull(J) = E_2$, we have $c\ell(I + J) = L^1(G)$. The question we are interested in can therefore be stated as: when do such ideals satisfy $I + J = L^1(G)$? In [91], H. Reiter defines the pair (E_1, E_2) to have the "decomposition property" with respect to $L^{\infty}(G)$ if for every $\varphi \in L^{\infty}(G)$ with $sp(\varphi) = F_1 \cup F_2$, F_i is a closed subset of E_i, $i = 1, 2$, there exists a decomposition of $\varphi = \varphi_1 + \varphi_2$ (necessarily unique) where $sp(\varphi_i) = F_i$, $i = 1, 2$. It is known that this is equivalent to the property: $I + J = L^1(G)$ for every pair of closed ideals in $L^1(G)$ with $hull(I) \subseteq E_1$ and $hull(J) \subseteq E_2$. We have this latter "closure property" obtained whenever one of the sets E_1 is compact as consequence of Wiener's Theorem (I, §4, Fact 11). Furthermore, these properties are not always satisfied as can be seen by the following example.

<u>Example</u>

Let $G = T$, the unit circle. Consider the function $f \in L^{\infty}(T)$ defined as

$$f(x) = \sum_{n=2}^{\infty} \frac{\sin(nx)}{n\ \ell og(n)} .$$

Let $E_1 = \{n \in Z : n < 0\}$ and $E_2 = \{n \in Z : n \geq 0\}$. Suppose f_1 and f_2 are in $L^\infty(T)$ with $sp(f_i) \subseteq E_i$, $i = 1, 2$ and $f = f_1 + f_2$. Then $\hat{f}(n) = \hat{f}_1(n) + \hat{f}_2(n) = \hat{f}_2(n)$ if $n \geq 0$. Therefore,

$$\sum_{n=2}^{\infty} \hat{f}_2(n)e^{inx} = \sum_{n=2}^{\infty} \hat{f}(n)e^{inx} = \sum_{n=2}^{\infty} \frac{1}{2in\ \ell og(n)}\ e^{inx}.$$

This is the Fourier series of an unbounded function in $L^1(T)$ [109, V, §1 and p. 253]. Hence, we have a contradiction and (E_1, E_2) does not have the decomposition property with respect to $L^\infty(T)$.

Definition 3.5. Let B be a Banach A-module, E_1 and E_2 be non-empty disjoint subsets of $\Delta(A)$. The pair (E_1, E_2) has

(i) the closure property with respect to A if for each pair of closed ideals I_1 and I_2 of A with $hull(I_i) \subseteq E_i$, $i = 1, 2$, we have $I_1 + I_2 = A$;

(ii) the decomposition property with respect to B if each $b \in B$ with $sp(b) = F_1 \cup F_2$, F_i a closed subset of E_i, $i = 1, 2$, has a decomposition (necessarily unique) $b = b_1 + b_2$, where $b_i \in B$ and $sp(b_i) = F_i$, $i = 1, 2$.

It is not difficult to see that our definition 3.5 extends that given by Reiter [91]. In Chapter IV, §5, we extend some of Reiter's results in order to progress toward an answer to the following problem.

CLOSURE PROBLEMS: Under what conditions does the closure or decomposition property hold for a Banach A-module B?

Conditions under which the closure and decomposition properties are equivalent as well as sufficient conditions to assure these properties will be provided in the next chapter.

Having defined a spectral synthesis problem and two related problems, we have reached one of our objectives: to show that spectral considerations in a Banach module context is possible. We now proceed to confirm that such considerations unify existing theories and serve to provide insight to the overall problems.

§6. Bi-annihilation Invariance

The vital role that duality plays in the study of spectral synthesis of bounded functions is quite evident. In particular, the Hahn-Banach Theorem is essential. This is apparent in the relation between L^1-spectral synthesis and L^∞-spectral synthesis where a closed subset of $\hat{G}$ is of L^1-spectral synthesis if and only if it is of L^∞-spectral synthesis (recall Chapter II).

A "duality condition" for Banach modules is introduced which allows us to investigate the problems in III, §5, in the spirit of the spectral synthesis of bounded functions. Furthermore, the condition as motivated by lemma 3.1 and the example succeeding it have interesting consequences in relation to almost periodicity and the norm topology (V, §4-§6).

The convention in III, §5, regarding the "τ-topology" is presumed to hold for the remainder of the text.

<u>Definition 3.6</u>. A Banach A-module B is <u>τ-bi-annihilation invariant</u> if the following two conditions are satisfied:

<u>HB1</u> $I^{\perp_B \perp_A} = I$ for every closed ideal of A; and

<u>HB2</u> $M^{\perp_A \perp_B} = M$ for every τ-closed submodule of B.

We shall write "B is a bi-annihilation invariant A-module" to mean that B is <u>norm</u>-bi-annihilation invariant.

Now for equivalent formulations of these conditions, we proceed to an easily verified proposition.

<u>Proposition 3.18</u>. Let B be a Banach A-module. Then

(a) Each of the following conditions is equivalent to <u>HB1</u>:

<u>HB1'</u> $I^{\perp_B} = J^{\perp_B}$ implies $I = J$ for all closed ideals I and J of A;

<u>HB1"</u> For every closed ideal I and $a \notin I$, there exists a $b \in B$ such that $I*b = \{0\}$ but $a*b \neq 0$.

(b) Each of the following conditions is equivalent to <u>HB2</u>:

<u>HB2'</u> $M^{\perp_A} = N^{\perp_A}$ implies $M = N$ for all τ-closed submodules M and N of B;

<u>HB2"</u> For every τ-closed submodule and $b \notin M$, there exists an $a \in A$ such that $a*M = \{0\}$ but $a*b \neq 0$.

<u>Proof</u>: The proofs of (a) and (b) are similar. We prove only the equivalences in (b). Suppose <u>HB2</u> holds and that M and N are τ-closed submodules of B with $M^{\perp_A} = N^{\perp_A}$. Clearly, $M^{\perp_A\perp_B} = N^{\perp_A\perp_B}$. Now <u>HB2</u> implies $M = M^{\perp_A\perp_B} = N^{\perp_A\perp_B} = N$ and so <u>HB2'</u> holds. Let us now assume B satisfies <u>HB2'</u>. Suppose M is a τ-closed submodule of B with $b \notin M$. Clearly, $\overline{[b]}^{\tau} \not\subset M$ and so $\overline{[b]}^{\tau\perp_A} \neq M^{\perp_A}$ by <u>HB2'</u>. If $\left(\overline{[b]}^{\tau\perp_A}\right)^c \cap M^{\perp_A}$ is empty, then $M^{\perp_A} \subset \overline{[b]}^{\tau\perp_A}$. Annihilating we obtain $M^{\perp_A\perp_B} \supset (\overline{[b]}^{\tau})^{\perp_A\perp_B} \supset \overline{[b]}^{\tau}$. But $b \in \overline{[b]}^{\tau}$. Thus, $b \in M^{\perp_A\perp_B} \setminus M$ implies $M^{\perp_A\perp_B} \neq M$. Condition <u>HB2'</u> yields with lemma 3.0 that $M^{\perp_A\perp_B} = M$, a contradiction. Hence $\left(\overline{[b]}^{\tau\perp_A}\right)^c \cap M^{\perp_A}$ is non-empty which gives the condition <u>HB2</u> . Finally, let us assume B satisfies <u>HB2"</u>. Since

$M \subseteq M^{\perp_A \perp_B}$ for any τ-closed submodule, we need to show $M^{\perp_A \perp_B} \setminus M$ is empty. Suppose $b \in M^{\perp_A \perp_B} \setminus M$. By HB2", there is an $a \in M^{\perp_A}$ such that $a*b \neq 0$. But then $a \notin M^{\perp_A \perp_B \perp_A}$, a contradiction to the fact that $M^{\perp_A} = M^{\perp_A \perp_B \perp_A}$. Thus, HB2" implies HB2. Q.E.D.

The use of the labels HB in the conditions considered above is justified by proposition 3.18 which relates the conditions to the Hahn-Banach (density) Theorem. In particular, the conditions HB1" and HB2" are analytical as opposed to the equivalent conditions HB1 and HB2 which emphasize the algebraic aspect.

Remarks 3.18

Let B be a τ-bi-annihilation invariant A-module.

1. For any closed ideal I of A, $\mathrm{hull}(I) = \mathrm{sp}(I^{\perp_B})$.

2. There is a one-to-one correspondence between the closed ideals of A and the τ-closed submodules of B. To see this, we note that for a closed ideal I of A, $I^{\perp}$ is a τ-closed submodule, hence annihilation induces a map $I \to I^{\perp_B}$ from the collection of all closed ideals in A to the τ-closed submodules of B. The condition of τ-bi-annihilation invariance guarantees that the correspondence is one-to-one. Similarly, we have for each τ-closed submodule M of B, $M^{\perp_A}$ a closed ideal. Consequently, the induced map $M \to M^{\perp_A}$ is by the same reasoning a one-to-one correspondence.

3. The smallest τ-closed submodule containing a submodule M of B is $M^{\perp_A \perp_B}$. Similarly, if I is an ideal of A, then $I^{\perp_B \perp_A}$ is the smallest closed ideal containing I. In fact, we have that $M^{\perp_A \perp_B} = c\ell_B^{\tau}(M)$ for any submodule of B.

It is easy to see that HB1 and HB2 are equivalent for $A = B$.

However, these two conditions are generally independent of one another.

Example 3.18

It will be shown in IV, §2, that $C_o(G)$ is not a bi-annihilation invariant $L^1(G)$-module (G non-compact). Specifically, $C_o(G)$ does not satisfy HB1, however, we now show it does satisfy HB2. Suppose N is a closed submodule of $C_o(G)$ and $\varphi \in C_o(G) \setminus N$. By the Hahn-Banach Theorem, there is a measure $\mu \in \mathcal{M}(G)$ such that $\varphi * \mu(0) = 0$ but $\int \psi(x)d\mu(x) = 0$ for all $\psi \in N$. Thus, $\varphi * \mu$ is bounded, continuous and non-zero. By duality, there is an $f \in L^1(G)$ such that $\int_G (\varphi*\mu)(x)f(-x)dx \neq 0$. But then $\int_G \varphi(x)(\mu*f)(-x)dx \neq 0$. This entails $\varphi*\mu*f \neq 0$. However, $\psi^* * \mu = 0$ for all $\psi \in N$ implies $\psi^* * \mu * f = 0$ for all $\psi \in N$. Consequently, if we set $g = f*\mu$, then $g \in L^1(G)*\mathcal{M}(G) \subseteq L^1(G)$ satisfies $g*N = \{0\}$ and $g*\varphi \neq 0$, i.e., $C_o(G)$ satisfies HB2.

In the following proposition, we do not assume A4 or A5. That is, we do not assume $A*b = 0$ implies $b = 0$ nor that $a*B = 0$ implies $a = 0$.

Proposition 3.19. Let B be a τ-bi-annihilation invariant A-module, then $B^{\perp A} = \{0_A\}$ and $A^{\perp B} = \{0_B\}$.

Proof: Since $\{0_B\}^{\perp A} = A$ and $\{0_A\}^{\perp B} = B$, we have $B^{\perp A} = \{0_A\}^{\perp B \perp A}$ and $A^{\perp B} = \{0_B\}^{\perp A \perp B}$. Conditions HB1 and HB2, respectively entail $B^{\perp A} = \{0_A\}$ and $A^{\perp B} = \{0_B\}$. Q.E.D.

Hence, τ-bi-annihilation invariant modules are order free. In the event that the modules we consider satisfy HB1 (resp. HB2), the assumption A4 (resp. A5) is not made, but automatically satisfied. In other cases, the standing hypotheses A1-A6 are in effect. The significance of this property (A4 or A5) is evident in the literature, for example see Gulick-Liu-van Rooij [51], [52], [53], [54], and Larsen [76], or the earlier citation to Graven.

In regard to our "duality" condition, it is of interest to see Dieudonné [29] and Kaplansky [64]. Kaplansky's results on dual rings are similar to, and in fact, motivated some of the considerations presented in this and the next section. Bi-annihilation invariance is an analogue of the "dual ring" concept for modules.

Kaplansky defines a _dual ring_ to be a topological ring A for which $R(L(I)) = I$ and $L(R(J)) = J$ for every closed right ideal I and closed left ideal J in A, where $L(I) = \{x \in A : xI = \{0\}\}$ and $R(J) = \{y \in A : Jy = \{0\}\}$ are the left and right annihilators, respectively. Thus, it is a consequence of the definition of dual ring that a one-to-one correspondence exists between the closed right and closed left ideals of A. It follows that $R(L(I))$ is the smallest closed right ideal containing I and $L(I) \cup L(J) = c\ell(L(I) + L(J)) = L(I \cap J)$ (see [64, §2]). We have the analogous assertions in propositions 3.24 and Remarks 3.8 (2-3). The next result shows that _bi-annihilation invariance agrees with the notion of "dual ring" in the case of commutative Banach algebras_.

Proposition 3.20. Let A be a commutative Banach algebra. If A is a dual ring, then A is bi-annihilation invariant A-module, and conversely.

Proof: Observe that the ideals of A are the same as the submodules of A regarded as a module over itself. Commutativity entails

$$L(I) = R(I) = I^{\perp_A} \text{ for a closed ideal of } A.$$

Therefore $L^{\perp A \perp A} = L(R(I)) = R(L(I))$ for all closed ideals of A and the conclusion follows. Q.E.D.

As a consequence of proposition 3.20, any dual ring which is a commutative Banach algebra serves as an example of a bi-annihilation invariant module.

Examples 3.20 (Kaplansky [64, §7])

1. Let G be a compact abelian group. The $L^p(G)$-spaces, $1 \leq p \leq \infty$, are dual rings, and hence, bi-annihilation invariant as L^p-modules. The convolution algebra $C(G)$ is also bi-annihilation invariant as an $C(G)$-module. In fact, any Segal algebra (III, §4, 2) $S(G)$ is a bi-annihilation invariant $S(G)$-module.

2. Let $A = B$ be an H^*-algebra satisfying A4 then B is a bi-annihilation invariant A-module.

Our next objective is to provide examples of $*$-bi-annihilation invariant modules. We do so by considering the dual spaces of certain algebras. Recall I, §5, example 4: if A is a commutative Banach algebra, A^* is an A-module with respect to the operation "$\circledast$" defined by the relation

$$(a, a_1 \circledast b) = (a_1 a, b) \quad \text{for all } a_1, a \in A \text{ and } b \in A^*.$$

For the set $\{b \in B : (a, b) = 0 \text{ for all } a \in I\}$ we write $I_\perp$ and similarly for $\{a \in A : (a, b) = 0 \text{ for all } b \in M\}$, $M_\perp$.

Lemma 3.21. Let $B = A^*$ be regarded as a Banach A-module with respect to $\circledast$. For any ideal I of A and any submodule M of B,

$$I^{\perp B} = I_\perp \quad \text{and} \quad M^{\perp A} = M_\perp .$$

Proof: Suppose $b \in I^{\perp}$ and $a_1 \in I$, then $a_1 \circledast b = 0$. Then $(a, a_1 \circledast b) = 0$ for all $a \in A$ and so $(a_1 a, b) = 0$ for all $a \in A$. Since A has an approximate identity, $a_1 \in \overline{a_1 A}$, therefore $(a_1, b) = 0$. This means $b \in I_{\perp}$ and we have $I^{\perp} \subseteq I_{\perp}$. For the opposite inclusion, let $b \in I_{\perp}$. For every $a \in A$ and $a_1 \in I$, $a_1 a \in I$, so that $(a_1 a, b) = 0$. By the definition of $\circledast$, we have $(a, a_1 \circledast b) = 0$ for all $a \in A$. This entails $a_1 \circledast b = 0$. Therefore, $b \in I^{\perp}$ and the equality $I^{\perp_B} = I_{\perp}$ is obtained. The proof of $M^{\perp_A} = M_{\perp}$ is similar and hence we omit it. Q.E.D.

We are now able to prove that all dual modules are $*$-bi-annihilation invariant.

Proposition 3.22. Let $B = A^*$ be regarded as a Banach A-module with respect to the operation $\circledast$. Then B is a $*$-bi-annihilation invariant A-module.

Proof: Suppose I is a closed ideal of A, then $I^{\perp_A \perp_B} = (I_{\perp})_{\perp}$ by lemma 3.21. The Hahn-Banach Theorem implies $I_{\perp\perp} = I$ and so $I^{\perp_A\perp_B} = I$. Since $*$-closed submodules M of B are weak-star closed, the dual form of the Hahn-Banach Theorem entails $M_{\perp\perp} = M$. Lemma 3.21 implies $M^{\perp_A\perp_B} = (M_{\perp})_{\perp}$ and so $M^{\perp_A\perp_B} = M$. Thus, $B = A^*$ satisfies HB1 and HB2. Q.E.D.

Examples 3.22

1. Let G be a LCAG. Let $B = L^{\infty}(G)$ and $A = L^1(G)$. By proposition 3.22, B is a $*$-bi-annihilation invariant A-module. This example may serve as our primary model in subsequent work.

2. More generally, let A be a Banach convolution subalgebra of

$L^1(G)$ satisfying A1-A4, then $B = A^*$ is a $*$-bi-annihilation invariant A-module (see examples 2-6, 9, in III, §4). Note that the assumptions A1-A4 are required as can be seen by the case G compact and $A = L^\infty(G)$. In this situation, $A^* = (L^\infty(G))^*$ does not satisfy A5.

3. Let G be a compact abelian group. The $L^p(G)$-algebras $(1 \leq p < \infty)$ are β-bi-annihilation invariant $L^1(G)$-modules. To see this, recall I, §6, Fact 4. The compactness of G entails the β-topology coincides with the bounded weak-star topology on $L^p(G)$ induced by $L^q(G)$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$. Thus, the Hahn-Banach Theorem yields the conclusion. Moreover, since the β-topology is stronger than the $*$-topology, any $*$-closed submodule is β-closed. Hence, the $L^p(G)$-algebras are $*$-bi-annihilation invariant as well [we note that, in general, if B is τ_1-bi-annihilation invariant and τ_1 is stronger than τ_2, then B is also τ_2-bi-annihilation invariant]. In the next section, we provide examples of norm-bi-annihilation invariance. The $L^p(G)$ - algebras are, in fact, norm bi-annihilation invariant $L^1(G)$-modules for G a compact abelian group.

§7. Some Structural Properties

Having introduced the concept of "τ-bi-annihilation invariance," we determine some structural properties of such modules. As the final considerations of this chapter, we attempt to provide insight into this "duality" condition by investigating some of the structure possessed by this class of modules.

First, we make a simple, but useful, observation which we record as a lemma.

Lemma 3.23. Let B_1 and B_2 be Banach A-modules with $B_2 \subseteq B_1$. If I is an ideal of A, then $I^{\perp B_2} = c\ell_{B_1}^{\tau}(I^{\perp B_2}) \cap B_2$.

Proof: Clearly, $I^{\perp B_2} \subseteq I^{\perp B_1}$. Now

$$c\ell_{B_1}^{\tau}(I^{\perp B_2}) \cap B_2 \subseteq c\ell_{B_1}^{\tau}(I^{\perp B_1}) \cap B_2 = I^{\perp B_1} \cap B_2 = I^{\perp B_2} \subseteq c\ell_{B_1}^{\tau}(I^{\perp B_2}) \cap B_2,$$

so that equality obtains throughout. Q.E.D.

Remark 3.23

The conclusion of lemma 3.23 resembles that of the ideal theorem for Segal algebras [92, 6, §2] or A-Segal algebras [14]. However, the conclusion in 3.23 is weaker since it requires the ideals (submodules) to be annihilators of submodules (ideals).

We next observe an interesting feature concerning τ-bi-annihilation invariant modules which will be applicable in our later work. In addition, the case $\tau \equiv$ norm topology extends the result for dual rings to modules [64, p. 690].

Let us denote the smallest τ-closed submodule containing $M \cup N$ by $\langle M \cup N \rangle$. Similarly, $\langle I \cup J \rangle$ denotes the smallest closed ideal containing $I \cup J$.

Proposition 3.24. Let B be a τ-bi-annihilation invariant A-module. If I and J are closed ideals of A, M and N are τ-closed submodules of B, then

(i) $(I \cap J)^{\perp B} = c\ell_B^{\tau}(I^{\perp B} + J^{\perp B})$; and

(ii) $(M \cap N)^{\perp A} = c\ell(M^{\perp A} + N^{\perp A})$.

Proof: Observe that $c\ell_B^{\tau}(I^{\perp B} + J^{\perp B})$ is the smallest τ-closed submodule containing $I^{\perp B} \cup J^{\perp B}$, namely $\langle I^{\perp B} \cup J^{\perp B}\rangle$. Similarly, $c\ell(M^{\perp A} + N^{\perp A}) = \langle M^{\perp A} \cup N^{\perp A}\rangle$. To prove the assertions, we first verify the relation

$$(*) \qquad (I^{\perp B} \cup J^{\perp B})^{\perp A} = I \cap J.$$

Since $I^{\perp B} \cup J^{\perp B}$ contains $I^{\perp B}$ and $J^{\perp B}$, we have $(I^{\perp B} \cup J^{\perp B})^{\perp A} \subseteq I^{\perp B \perp A} \cap J^{\perp B \perp A}$. Thus, $(I^{\perp B} \cup J^{\perp B})^{\perp A} \subseteq I \cap J$ by HB1. For the reverse inclusion, suppose $a \in I \cap J$. If $b \in I^{\perp B} \cup J^{\perp B}$, then $a*b = 0$. Therefore, $a \in (I^{\perp B} \cup J^{\perp B})^{\perp A}$ and the equality (*) follows. By annihilating (*), we obtain $(I \cap J)^{\perp B} = (I^{\perp B} \cup J^{\perp B})^{\perp A \perp B}$. But $(I^{\perp B} \cup J^{\perp B})^{\perp A} = \langle I^{\perp B} \cup J^{\perp B}\rangle^{\perp A}$ so that $(I \cap J)^{\perp B} = \langle I^{\perp B} \cup J^{\perp B}\rangle^{\perp A \perp B} = \langle I^{\perp B} \cup J^{\perp B}\rangle$. This is (i), the proof of (ii) is analogous. Q.E.D.

Remark 3.24

One must be cautious in interpreting the conclusions of proposition 3.24. That is, annihilation does not "act" like complimentation since for example $(I \cap J)^{\perp} = c\ell(I^{\perp} + J^{\perp}) \supseteq I^{\perp} \cup J^{\perp}$ and generally $I^{\perp} \cup J^{\perp}$ is properly contained in $\langle I^{\perp} \cup J^{\perp}\rangle$.

Our next result is an extension of Kaplansky [64, Th. 1, pg. 609].

Proposition 3.25. Let B be a τ-bi-annihilation invariant A-module. If $b \in B$, then $b \in c\ell_B^{\tau}(A*b)$.

Proof: Observing that $[b]^{\perp A} = c\ell_B^{\tau}(A*b)^{\perp A}$, we have that $b \in b^{\perp A \perp B} = [b]^{\perp A \perp B} = c\ell_B^{\tau}(A*b)^{\perp A \perp B} = c\ell_B^{\tau}(A*b)$. Q.E.D.

Corollary 3.26. τ-bi-annihilation invariant modules have "τ-approximate identities." In particular, if B is bi-annihilation invariant, then B is essential $(B = \overline{A*b})$.

A question which arises naturally is "what submodules of a -bi-annihilation invariant submodule are τ-bi-annihilation invariant?" In fact, it is of interest to examine some properties of submodules in τ-bi-annihilation invariant modules to better understand the concept. Consequently, our objective in the remainder of this section is to determine some structural conditions on submodules to insure τ-bi-annihilation invariance.

<u>Theorem 3.27</u>. Let B_1 and B_2 be Banach A-modules. Suppose $B_2 \subset B_1$ and B_1 is τ-bi-annihilation invariant. If I is a closed ideal of A, then $I^{\perp B_2}$ is τ-dense in $I^{\perp B_1}$ if and only if $I = I^{\perp B_2 \perp A}$.

<u>Proof</u>: Since B_1 is τ-bi-annihilation invariant, $I^{\perp B_2 \perp A \perp B_1} = c\ell^{\tau}_{B_1}(I^{\perp B_2})$. Thus, if $I = I^{\perp B_2 \perp A}$, then $I^{\perp B_1} = I^{\perp B_2 \perp A \perp B_1} = c\ell^{\tau}_{B_1}(I^{\perp B_2})$.

For the forward implication, suppose $I^{\perp B_2}$ is τ-dense in $I^{\perp B_1}$. The above relation entails $I^{\perp B_1} = c\ell^{\tau}_{B_1}(I^{\perp B_2}) = I^{\perp B_2 \perp A \perp B_1}$. Annihilating, we obtain $I^{\perp B_1 \perp A} = I^{\perp B_2 \perp A \perp B_1 \perp A}$. By lemma 3.0 and τ-bi-annihilation invariance we have $I = I^{\perp B_1 \perp A} = I^{\perp B_2 \perp A}$.

Q.E.D.

<u>Corollary 3.28</u>. Let B_1 and B_2 be Banach A-modules. Suppose $B_2 \subseteq B_1$ and B_1 is τ-bi-annihilation invariant. If B_2 is τ-bi-annihilation invariant, then B_2 is τ-dense in B_1.

<u>Proof</u>: For $I = \{0_A\}$, $I^{\perp B_2 \perp A} = I$ since $B_2^{\perp A} = \{0\}$ by proposition 3.19. Theorem 3.26 entails $I^{\perp B_2}$ is τ-dense in $I^{\perp B_1}$. But $I^{\perp B_1} = B$ and $I^{\perp B_2} = B_2$ which yields that B_2 is τ-dense in B_1.

Q.E.D.

Remarks 3.28

1. The Corollary 3.27 can be proven directly by observing that $c\ell^{\tau}_{B_1}(B_2) = B_2^{\perp A \perp B_2}$ but we prefer it as an easy consequence of Theorem 3.27 which itself provides insight into the properties of τ-bi-annihilation invariance.

2. Evidently, there are <u>no proper τ-closed submodules</u> of B, a τ-bi-annihilation invariant A-module, <u>which are themselves τ-bi-annihilation invariant</u>. This does not exclude the possibility of a τ_1-closed submodule of a τ_2-bi-annihilation invariant module being a τ_2-bi-annihilation invariant module. For instance, $C(G)$, G compact, is $*$-dense and a $*$-bi-annihilation invariant submodule of $L^{\infty}(G)$, but norm closed.

3. <u>The converse of Corollary 3.28 is false</u>. Although denseness is necessary, it is not sufficient for a submodule to be τ-bi-annihilation invariant. As an example, consider $C_o(G)$, G a non-compact LCAG. $C_o(G)$ is $*$-dense in $L^{\infty}(G)$ (in fact, β-dense), but as remarked in example 3.18, $C_o(G)$ is not $*$-bi-annihilation invariant.

Remark 3.28 (3) prompts us to find sufficient conditions for submodules of τ-bi-annihilation invariant modules to be τ-bi-annihilation invariant. <u>We now present our main result characterizing τ-bi-annihilation invariant submodules</u>.

<u>Theorem 3.29</u>. Let B_1 be a τ-bi-annihilation invariant A-module. Let B_2 be a submodule of B_1, then the following are equivalent:

(i) B_2 is τ-bi-annihilation invariant;

(ii) $M = c\ell^{\tau}_{B_1}(M) \cap B_2$ for every τ-closed submodule M of B_2 and $I^{\perp B_2}$ is τ-dense in $I^{\perp B_1}$ for every closed ideal I of A.

Proof: (i) $\Longrightarrow$ (ii) Suppose B_2 is τ-bi-annihilation invariant, then Theorem 3.26 entails $I^{\perp B_2}$ is τ-dense in $I^{\perp B_1}$ for every closed ideal I of A by HB1. For a τ-closed submodule M of B_2,

$$M = M^{\perp A \perp B_2} = M^{\perp A \perp B_1} \cap B_2 = c\ell^{\tau}_{B_1}(M) \cap B_2.$$

(ii) $\Longrightarrow$ (i) Suppose the conditions in (ii) are satisfied. Let M be a τ-closed submodule of B_2. Then

$$M^{\perp A \perp B_2} = M^{\perp A \perp B} \cap B_2 = c\ell^{\tau}_{B_1}(M) \cap B_2 = M.$$

For a closed ideal I of A,

$$I^{\perp B_2 \perp A} = \left[c\ell^{\tau}_{B_1}(I^{\perp B_2})\right]^{\perp A} = I^{\perp B_1 \perp A} = I.$$

Therefore, B_2 is τ-bi-annihilation invariant. Q.E.D.

Remarks 3.29

1. It appears as though the condition "$I^{\perp B_2}$ is τ-dense in $I^{\perp B_1}$" is difficult to weaken, for instance, if $I = I_{\hat{x}}$, then $I^{\perp C_o} = \{0\}$ (recall A7) and yet $L^{\perp L^\infty} \neq \{0\}$ so that $I^{\perp C_o}$ is not τ-dense in $L^{\perp L^\infty}$. As we have mentioned, C_o is not τ-bi-annihilation invariant.

2. The condition "$M = c\ell^{\tau}_{B_1}(M) \cap B_2$" for τ-closed submodules of B_2 is reminiscent of the Ideal Theorem for Segal algebras and their generalizations. In particular, we cite Reiter's Fundamental Theorem on Segal algebras [91, p. 129] as well as Burnham's study of Abstract Segal

algebras [14]. Wildfogel preys on such a result in his study of "Double algebras" [108]. The demand for such a condition is evident by our context as well.

We now proceed to an application.

Corollary 3.30. Let G be a compact abelian group. Any Segal algebra is a bi-annihilation invariant $L^1(G)$-module.

Proof: We apply Theorem 3.28 with $A = B_1 = L^1(G)$ and $B_2 = S(G)$, a Segal algebra. By example 3.20 (1), $L^1(G)$ is a bi-annihilation invariant $L^1(G)$-module. It is known $S(G)$ is an $L^1(G)$-module (III, §4, 2). The Fundamental Theorem for Segal algebras states that for any closed ideal $I_s \subseteq S(G)$, $I_s = c\ell_{L^1}(I_s) \cap S(G)$. Since closed submodules M of $S(G)$ are, in fact, closed ideals of $S(G)$, we have $M = c\ell_{L^1}(M) \cap S(G)$. Now for a closed ideal I of $L^1(G)$, $I^{\perp S}$ is a closed submodule of $S(G)$. Therefore, $I^{\perp S} = c\ell_{L^1}(I^{\perp S}) \cap S(G)$, uniquely. But since $I^{\perp S} = I^{\perp L^1} \cap S(G)$, we have by uniqueness that $c\ell_{L^1}(I^{\perp S}) = I^{\perp L^1}$.

Q.E.D.

The following result provides sufficient conditions for a Banach module to be τ-bi-annihilation invariant in case it contains a τ-bi-annihilation invariant submodule.

Theorem 3.31. Let B_1 be a Banach A-module. Suppose B_2 is a τ-bi-annihilation invariant submodule of B_1, and that the following conditions are satisfied:

(i) $M \cap B_2$ is τ-dense in M for every τ-closed submodule M of B_1;

(ii) $M \cap B_2$ is a τ-closed submodule of B_2 for every τ-closed submodule of B_1, then

B is a τ-bi-annihilation invariant A-module.

Proof: Suppose I is a closed ideal of A. Since $B_2 \subseteq B_1$, $I^{\perp B_2} \subseteq I^{\perp B_1}$. By the properties of annihilators and HB1 $I^{\perp B_1 \perp A} \subseteq I^{\perp B_2 \perp A} = I$. But $I \subseteq I^{\perp B_1 \perp A}$ always obtains so that $I^{\perp B_1 \perp A} = I$. Now let M be a τ-closed submodule of B_1. By (i), (ii), and HB2,

$$M \cap B_2 = (M \cap B_2)^{\perp A \perp B_2} = \left[c\ell^{\tau}_{B_1}(M \cap B_2)\right]^{\perp A \perp B_2} = M^{\perp A \perp B_2}.$$

By (i), $c\ell^{\tau}_{B_1}(M^{\perp A \perp B_2}) = c\ell^{\tau}_{B_1}(M \cap B_2) = M$. Since

$$M^{\perp A \perp B_2} = M^{\perp A \perp B_1} \cap B_2,$$

we have

$$M = c\ell^{\tau}_{B_1}(M^{\perp A \perp B_2}) = c\ell^{\tau}_{B_1}(M^{\perp A \perp B_1} \cap B_2).$$

Applying (i) to $M^{\perp A \perp B_1}$, we obtain $M = M^{\perp A \perp B_1}$. Q.E.D.

An analogous theorem is obtained by alteration of the algebra. That is, conditions are obtained to determine when a τ-bi-annihilation invariant A_1-module B is also a τ-bi-annihilation invariant A_2-module if $A_2 \subseteq A$.

Theorem 3.32. Let A_1 and A_2 be Banach algebras with A_2 a subalgebra of A. Suppose B is a τ-bi-annihilation invariant A_1-module, then B is a τ-bi-annihilation invariant A_2-module if and only if

(i) $I = c\ell_{A_1}(I) \cap A_2$ for every closed ideal of A_2, and

(ii) $M^{\perp A_2}$ is dense in $M^{\perp A_1}$ for every τ-closed submodule of B.

Proof: Suppose B is a τ-bi-annihilation invariant A_2-module. Since $M^{\perp A_1} = (M^{\perp A_2 \perp B})^{\perp A_1} = (M^{\perp A_2})^{\perp B \perp A_1} = c\ell_{A_1}(M^{\perp A_2})$, we have (ii). Also, for I a closed ideal of A_2.

$$I \subseteq c\ell_{A_1}(I) \cap A_2 = I^{\perp B \perp A_1} \cap A_2 = I^{\perp B \perp A_2}$$

and (i) follows.

For the other implication, suppose (i) and (ii) hold. Let $I \subseteq A_2$ be a closed ideal. Now

$$I^{\perp B \perp A_2} \subseteq I^{\perp B \perp A_1} \subseteq \left[c\ell_{A_1}(I)\right]^{\perp B \perp A_1} = c\ell_{A_1}(I).$$

Applying (i),

$$I = c\ell_{A_1}(I) \cap A_2 \supseteq I^{\perp B \perp A_2} \cap A_2 = I^{\perp B \perp A_2}.$$

Since the opposite inclusion always holds, we obtain HB1. Now let M be any τ-closed submodule of B. Condition (ii) entails

$$M^{\perp A_2 \perp B} = \left[c\ell_{A_1}(M^{\perp A_2})\right]^{\perp B} = M^{\perp A_1 \perp B} = M.$$

Therefore, B is a τ-bi-annihilation invariant A_2-module. Q.E.D.

We see in (i) of Theorem 3.31 the condition alluded to in Remark 3.39 (1). The impact of algebra is apparent in our considerations thus-far, and indeed, this persists as part of our theme.

CHAPTER IV

ELEMENTARY SPECTRAL SYNTHESIS

The present chapter undertakes the task of establishing basic results in the spectral synthesis of Banach modules. The first section utilizes the concept of τ-bi-annihilation invariance to relate the spectral synthesis theory in Banach algebras to our theory in Banach modules. A characterization of sets of spectral synthesis is obtained for τ-bi-annihilation invariant modules (Theorem 4.1). We then are able to provide conditions under which spectral synthesis is valid. The next section provides the definition of "sets of multiplicity" in the context of modules and consequences. An examination of elements with one-point spectra is then made. Here, we relate to a problem of Reiter. In Chapter V, the elements of one-point spectra play an interesting role (V, §4-5). This chapter concludes with investigations into the closure problem in modules.

We are able to enhance our proposition of viewing spectral synthesis problems in the context of Banach modules by sustaining a basis for a theory in this chapter. In particular, we unify results for "particulars" and transfer them into the module context for a different perspective--and perhaps better understanding.

§1. Spectral Synthesis and τ-Bi-annihilation Invariance

The first result is a characterization of S-B sets. Recalling Fact 5 of Chapter I, §3, we are not surprised, but gratified, that our own result obtains in the context of modules.

It is easy to conclude that for an S-A set E, $I(E)^{\perp A} = J(E)^{\perp A}$. Our characterization provides conditions under which E is then also an S-B set. The case $A = L^1(G)$ and $B = L^\infty(G)$ is well-known in the theory of weak-star spectral synthesis ([61, §40]).

<u>Theorem 4.1</u>. Let B be a τ-bi-annihilation invariant A-module and E a closed subset of $\Delta(A)$. The following are equivalent:

(i) E is an S-A set;

(ii) $I(E) = J(E)$;

(iii) $b \in B$ with $sp(b) \subseteq E$ implies $a*b = 0$ for all $a \in I(E)$;

(iv) E is an S-B set.

<u>Proof</u>: The equivalence of (i) and (ii) is a restatement of part of the conclusion of Fact 5, I, §3, which is stated here for completeness.

(ii)$\Longrightarrow$(iv) Suppose M is a τ-closed submodule with $sp(M) = E$. Then $hull(M^\perp) = E$ and (ii) implies $M^\perp = I(E) = J(E)$. Thus, $M^{\perp\perp} = I(E)^\perp = J(E)^\perp$, and <u>HB2</u> entails $M = I(E)^\perp = J(E)^\perp$. So M must be unique, and hence, E is an S-B set.

(iv)$\Longrightarrow$(ii) Since for a closed ideal I of A, $sp(I^\perp) = hull(I^{\perp\perp}) = hull(I) = E$, we have $I(E)^\perp = J(E)^\perp$. Condition <u>HB1</u> implies $I(E) = I(E)^{\perp\perp} = J(E)^{\perp\perp} = J(E)$.

(iii)$\Longrightarrow$(iv) This is evident by the fact that (iii) can be expressed as $J(E)^{\perp B} \subseteq I(E)^{\perp B}$; together with $I(E) \supseteq J(E)$ implying $I(E)^{\perp B} \subseteq J(E)^{\perp B}$, and hence $I(E)^{\perp B} = J(E)^{\perp B}$. Now τ-bi-annihilation invariance applies to give E is an S-B set if and only if $I(E)^{\perp B} = J(E)^{\perp B}$. Q.E.D.

Theorem 4.1 provides insight into the validity of spectral synthesis in modules, and also exhibits the potentiality of τ-bi-annihilation invariance. We now obtain a <u>negative</u> result concerning an important class of algebras.

<u>Proposition 4.2</u>. Let G be a non-compact LCAG. Any Segal algebra $S(G)$ regarded as an $L^1(G)$-module is non-bi-annihilation invariant.

<u>Proof</u>: By Malliavin's Theorem (I, §4, Fact 13), there is a closed subset E of $\hat{G}$ which is not a set of spectral synthesis relative to $L^1(G)$. Therefore, $I(E) \neq J(E)$. Consider $I(E)^{\perp S}$ and $J(E)^{\perp S}$. Clearly, $I(E)^{\perp S} \subseteq J(E)^{\perp S}$. Suppose $\varphi \in J(E)^{\perp S}$, then $sp(\varphi) \subseteq E$. But $\varphi \in S(G) \subseteq L^1(G)$ so that $sp(\varphi) = \sigma(\varphi)$. Now for $f \in I(E)$, $\hat{f}(E) = 0$, and this entails $\sigma(f) \subseteq c\ell(E^c)$. But then $f * \varphi \in S(G)$ is such that $\widehat{f*\varphi}(\hat{x}) = \hat{f}(\hat{x})\hat{\varphi}(\hat{x}) = 0$ for all $\hat{x} \in \hat{G}$. Therefore, $f * \varphi = 0$ and $\varphi \in I(E)^{\perp S}$. Consequently, $J(E)^{\perp S} = I(E)^{\perp S}$. Since E is <u>not</u> an S-A set, the equivalence (i)$\Leftrightarrow$(iii) of Theorem 4.1 does not hold. The $L^1(G)$-module $S(G)$ is non-bi-annihilation invariant. Q.E.D.

<u>Remarks 4.2</u>

1. Proposition 4.2 entails that a Banach A-module B can satisfy: a closed subset E of $\Delta(A)$ is an S-A set if and only if E is an S-B set, yet be non-bi-annihilation invariant. In particular, Segal algebras (or $L^1(G)$-modules) have the same S-sets as $L^1(G)$ [15].

2. An easy corollary of Proposition 4.2 is that for any Segal algebra $S(G)$, G a <u>compact</u> group, spectral synthesis is valid. But

this is known from the ideal theory of Segal algebras. In particular, we have another indication of why compactness is essential in example 3.31 of §6, Chapter III.

A consequence of Theorem 4.1 is that it provides insight into the consideration of spectral synthesis in general, as does our approach in the context of modules. In fact, this avenue has not been utilized to its complete potential. With the exception of the work by Domar [30], [32], Kitchen [69], and to a lesser extent Dunkl [34], a spectral synthesis theory in modules has not been exploited. We provide an illustration at this point in the form of a conjecture.

Conjecture 4.3

K. deLeeuw and H. Mirkil considered a spectral synthesis problem for the space $C_o(G)$ with the relative weak-star topology induced by $L^1(G)$ [23]. They query whether two given uniformly closed translation invariant subalgebras M_1 and M_2 of $C_o(G)$ with $sp(M_1) = sp(M_2)$ are equal. Furthermore, they state that this spectral synthesis problem "seems to be logically distinct from the famous one solved by Malliavin." With the insight we have gained up to now, we feel that this is not the case. Let us consider the "two" forms of spectral synthesis:

(1) Given two closed submodules M and N, does $sp(M) = sp(N)$ imply $M = N$?

(2) Given two closed ideals I and J of A, does $hull(I) = hull(J)$ imply $I = J$?

Theorem 4.1 shows that if B satisfies HB1 and (1) is true, then (2) is true; and also that if B satisfies HB2 and (2) is true, then (1) is true. For τ-bi-annihilation invariant modules we see that the problems (1) and (2) are indeed logically equivalent, and hence for τ-bi-annihilation invariant $L^1(G)$-modules, (1) is equivalent to the problem solved by Malliavin. For Banach $L^1(G)$-modules satisfying <u>HB1</u>, there exists closed submodules M and N for which (1) is negative. Unfortunately, $C_o(G)$ is a Banach $L^1(G)$-module satisfying <u>HB2</u> and not <u>HB1</u> so that we are unable to prove the equivalence of (1) and (2) in this case. However, <u>we do conjecture that the query by de Leeuw and Mirkil is to be answered negatively</u>. <u>That is, $sp(M) = sp(N)$ does not imply $M = N$</u>. The discussion of spectral synthesis in Banach modules to this point appears to bear this out. Only further investigation will resolve the validity of our conjecture--for instance, obtaining Theorem 4.1 for a larger class of modules which includes, say, all essential $L^1(G)$-modules.

§2. Sets of Multiplicity and Submodules

We consider a concept which allows us to determine when a closed set in $\Delta(A)$ determines a submodule of a Banach A-module B. This concept--sets of multiplicity--possesses an interesting relationship with the annihilator submodules of closed maximal ideals of A.

The following definitions utilized by de Leeuw and Mirkil for the case $B = C_o(G)$, $A = L^1(G)$ [23] come, of course, from the classical theory of trigonometric series.

<u>Definition 4.1</u>. A subset E of $\Delta(A)$ is a <u>set of multiplicity</u> if there exists a non-zero b in B with $sp(b) \subseteq E$.

A subset E of $\Delta(A)$ is <u>locally a set of multiplicity</u> if $U \cap E$

is a set of multiplicity for each open set U which satisfies $U \cap E \neq \phi$.

Remarks 4.3

1. Since $U \cap E \subseteq E$, we obviously have that E is a set of multiplicity if it is locally a set of multiplicity (cf. example 4.3).

2. For a set $E \subseteq \Delta(A)$, $\overline{E}$ is of multiplicity if and only if $J(\overline{E})^{\perp B} \neq \{0\}$. In fact, if E is of multiplicity, $\overline{E}$ is of multiplicity and $J(\overline{E})^{\perp B} \neq \{0\}$.

3. For any non-zero b in B, $sp(b)$ is of multiplicity.

We point out that definition 4.1 agrees with the one used in the uniqueness theory of trigonometric series [109, p. 344] for $\Delta(A) = Z$. In that theory, E is an M-set (a set of multiplicity) if there is a trigonometric series converging outside E but is not identically zero.

It is not difficult to show that the converse of Remark 4.3 (1) is false.

Example 4.3

Let $f \in L^1(IR)$ such that Fourier transform is given by

$$\hat{f}(x) = \begin{cases} 1 & |x| \leq 1 \\ 2 - |x| & 1 \leq |x| \leq 2 \\ 0 & |x| \geq 2 \end{cases}$$

(cf. [92, p. 3]). Let $B = [f]$. Now B is a Banach $L^1(IR)$-module with respect to convolution. Moreover, for each $\varphi \in B$, $sp(\varphi) \subseteq sp([f]) = sp(f) = \sigma(f) = [-2,2]$. Let $E = [-2,3]$. Clearly, E is a set of multiplicity. For the open set $U = (2,3)$, $E \cap U \neq \phi$.

However, E cannot be locally of multiplicity since for each $\varphi \in B$, $sp(\varphi) \cap (E \cap U) = sp(\varphi) \cap U = \phi$. Therefore, the converse of Remark 4.3 (1) is false.

We now apply the concept of multiplicity to obtain necessary and sufficient conditions for a closed subset of $\Delta(A)$ to be the spectrum of a τ-closed submodule.

Theorem 4.4. Let B be a Banach A-module. A closed nonempty subset E of $\Delta(A)$ is locally a set of multiplicity if and only if there is a nontrivial τ-closed submodule M of B with $sp(M) = E$.

Proof: Suppose E is locally a set of multiplicity. Since $J(E)^{\perp B \perp A} \supseteq J(E)$, $sp(J(E)^{\perp B}) = hull(J(E)^{\perp B \perp A}) \subseteq hull(J(E)) = E$. But $J(E)^{\perp B}$ is a τ-closed submodule of B by 3.16. Thus we only need to show $E \subseteq sp(J(E)^{\perp B})$. Suppose $\hat{x} \in E$. Let U be a nbhd. of $\hat{x}$, then $U \cap E \neq \phi$. Now E locally a set of multiplicity implies there is a $b \in B$ such that $sp(b) \subseteq U \cap E$. Note then that $b \in J(E)^{\perp B}$ since $sp(b) \subseteq E$. Thus, U an arbitrary nbhd. of $\hat{x}$ entails the existence of a net $\langle \hat{x}_\alpha \rangle$ converging to $\hat{x}$ where for each α, $\hat{x}_\alpha \in sp(b_\alpha) \subset U_\alpha \cap E$, $b_\alpha \in B$, U_α a nbhd. of $\hat{x}$. Hence, $sp(J(E)^{\perp B})$ closed with $\langle \hat{x}_\alpha \rangle \subset sp(J(E)^{\perp B})$ converging to $\hat{x}$ implies $\hat{x} \in sp(J(E)^{\perp B})$. Consequently, $E = sp(J(E)^{\perp B})$.

On the other hand, let M be a τ-closed submodule of B with $sp(M) = E$. Suppose U is an open set with $U \cap E \neq \phi$. By proposition 3.10, $sp(M) = c\ell(\bigcup_{b \in M} sp(b))$ so that the normality of $\Delta(A)$ entails $U \cap (\bigcup_{b \in M} sp(b)) \neq \phi$. Now there is an $\hat{x} \in U \cap (\bigcup_{b \in M} sp(b))$, and consequently $\hat{x} \in U \cap sp(b_o)$ for some $b_o \in M$. Let $a \in A_c$ be such that $\hat{a}(\hat{x}) \neq 0$ and $\sigma(a) \subset U$. Set $b = a*b_o \neq 0$. Then $b \in M$ and $sp(b) \subseteq U \cap sp(M)$. Thus, $sp(M)$ is locally a set of multiplicity.

Q.E.D.

The concept of multiplicity applied to algebra modules leads to an extension of results in [23]. In particular, the spectrum of a τ-closed submodule is perfect.

Proposition 4.5. Let B be an algebra module with $B \subseteq L^1(G)$. Let E be locally a set of multiplicity. The set E does not contain any isolated points unless $\hat{G}$ is discrete.

Proof: Suppose $\hat{x} \in E$ is isolated. Now there is a nbhd. U of $\hat{x}$ satisfying $(E\setminus\{x\}) \cap U = \emptyset$. Since E is locally a set of multiplicity, there is a $\varphi \in B$ with $sp(\varphi) \subseteq \{\hat{x}\}$ and $\varphi \neq 0$. Since $B \subseteq L^1(G)$, we have $\sigma(\varphi) = \{\hat{x}\}$. Hence, $\varphi \equiv \alpha\,\hat{x}$ for some $\alpha \in \mathbb{C}$, and $\alpha\,\hat{x} \in L^1(G)$ forces G to be compact. Q.E.D.

An immediate consequence is the following.

Corollary 4.6. Let B be an algebra module with $B \subseteq L^1(G)$. Let M be a τ-closed submodule of B. Then $sp(M)$ is a perfect set.

Proof: Theorem 4.4 entails $sp(M)$ is locally a set of multiplicity. Proposition 4.5 together with the fact that $sp(M)$ is closed entails $sp(M)$ is perfect. Q.E.D.

The most noteworthy algebra modules to which the last two results apply are the Segal algebras.

Further structure of the spectra of submodules can be determined in the case of algebra modules. Our final two results of this section illustrate this fact.

Theorem 4.7. Let B be an algebra A-module and M a τ-closed submodule of B. The spectrum of M is a closed subsemigroup of $\hat{G}$.

Proof: Let $\hat{x}$ and $\hat{y}$ be in $sp(M)$. We need to show that $\hat{x} + \hat{y} \in sp(M)$. We first assume $\hat{x}$ and $\hat{y}$ are in $\bigcup_{b\varepsilon M} sp(b)$. For a symmetric 0-nbhd. U, $(\hat{x} + U) \cap (\bigcup_{b\varepsilon M} sp(b)) \neq \phi$ and $(\hat{y} + U) \cap (\bigcup_{b\varepsilon M} spb)) \neq \phi$. Thus, there are elements b and c in M with $sp(b) \subseteq \hat{x} + U$ and $sp(c) \subseteq \hat{y} + U$. Set $m = b + c$. By proposition 3.6 (iii), $sp(m) \subseteq sp(b) \cup sp(c)$. Therefore, $sp(m) \subseteq (\hat{x} + U) + (\hat{y} + U) = (\hat{x} + \hat{y}) + (U + U) \subseteq (\hat{x} + \hat{y}) + U$. But U is arbitrary (symmetric nbhds. form a basis for $\hat{G}$) and so $\hat{x} + \hat{y}$ is arbitrarily close to $sp(m)$. Since $m \varepsilon M$ and $sp(m) \subseteq sp(M)$ with $sp(M)$ closed, $\hat{x} + \hat{y}$ is in $sp(M)$.

To complete the proof, we observe $sp(M) = c\ell(\bigcup_{b\varepsilon M} sp(b))$. For $\hat{x}, \hat{y} \varepsilon sp(M)$, let $\langle \hat{x}_\alpha \rangle$ and $\langle \hat{y}_\alpha \rangle$ be in $\bigcup_{b\varepsilon M} sp(b)$ such that $\hat{x}_\alpha \to \hat{x}$ and $\hat{y}_\alpha \to \hat{y}$. By the first part of the proof, $\hat{x}_\alpha + \hat{y}_\alpha \varepsilon sp(M)$ for all α. Since $\hat{G}$ is a topological group we see that $\hat{x}_\alpha + \hat{y}_\alpha \to \hat{x} + \hat{y}$. The fact that $sp(M)$ is closed renders $\hat{x} + \hat{y} \varepsilon sp(M)$. Q.E.D.

Note that we only need that $\Delta(A)$ is a topological group, but we prefer to state the theorem in terms of algebra modules.

Theorem 4.8. Let B be an algebra A-module which is also an algebra. Let E be a closed subset of $\hat{G}$. If E is a subsemigroup of $\hat{G}$, then $J(E)^{\perp B}$ is a τ-closed submodule of B which is also an invariant subalgebra.

Proof: By our previous considerations, $J(E)^{\perp B}$ is a τ-closed submodule of B. We need to verify that it is an invariant subalgebra. Suppose b_1 and b_2 are in $J(E)^{\perp B}$. By proposition 3.17, $sp(b_1 b_2) \subseteq c\ell[sp(b_1) + sp(b_2)] \subseteq c\ell(E + E) = c\ell(E) = E$ where we use the fact that E is semigroup and proposition 3.14. By again applying 3.14, $sp(b_1 b_2) \subseteq E$

implies $b_1 b_2 \in J(E)^{\perp B}$. Moreover, $sp(b_1 * b_2) \subseteq sp(b_1) \cap sp(b_2) \subseteq E$. Thus, $b_1 * b_2 \in J(E)^{\perp B}$. For invariance, we apply proposition 3.17 to obtain $sp(L_x b) = sp(b)$ and $L_x b \in J(E)^{\perp B}$ for $b \in J(E)^{\perp B}$. Q.E.D.

§3. Elements with One-point Spectra

In this section we are interested in the elements of a Banach A-module B having a singleton as spectrum, that is, the elements $b \in B$ with $sp(b) \subseteq \{\hat{x}\}$ for some $\hat{x} \in \Delta(A)$. Analysis of these elements appear in the investigations of spectral synthesis by various authors, for example Domar [30], Herz [59] and Edwards [38].

Our purpose is to provide another link in the chain of relationships between the spectral synthesis in Banach algebras (and of bounded functions) and a theory of spectral synthesis in Banach modules. We do so by obtaining an easily derived characterization of elements with one-point spectra. This is the content of Theorem 4.9, and the remainder of the section is devoted to interpretation of the theorem and applications. The relationship between these elements and almost periodicity will be exhibited in the next chapter.

We point out that the following proof requires A6. Furthermore, the equivalence between (ii) and (iii) is known, but we incorporate it with our result for the sake of completeness (for example see [69]).

Theorem 4.9. Let B be a Banach A-module and $\hat{x} \in \Delta(A)$. If b is a nonzero element in B, then the following are equivalent:

(i) $sp(b) = \{x\}$;

(ii) $a * b = \hat{a}(\hat{x}) b$ for all $a \in A$;

(iii) $\dim([b]) = 1$.

If, in addition, B satisfies HB2, then each of (i)-(iii) is equivalent to

(iv) $[b] = \{b' \in B : a*b' = \hat{a}(\hat{x})b'$ for all $a \in A\}$.

<u>Proof</u>: We remark that $\dim([b])$ is the dimension of $[b]$ as a vector space over $\mathbb{C}$.

(i)$\Longrightarrow$(ii) Suppose $a \in A$ is arbitrary. Choose $\hat{a}_1 \in A$ such that $\hat{a}_1 \equiv 1$ on a relatively compact nbhd. of $\hat{x}$. Set $a_o = a - a_1\hat{a}(\hat{x})$. We have that $\hat{a}_o(\hat{x}) = 0$ and so $a_o \in I_{\hat{x}}$. By <u>A6</u>, $I_{\hat{x}} = J(\{\hat{x}\})$. This together with (i) entail that $a_o*b = 0$, i.e., $b \in J(\{\hat{x}\})^{\perp B}$. Equivalently, $(a - a_1\hat{a}(\hat{x}))*b = 0$ or $a*b = a_1\hat{a}(\hat{x})*b = \hat{a}(\hat{x})a_1*b = \hat{a}(\hat{x})b$ by proposition 3.6.

(ii)$\Longrightarrow$(iii) Suppose $a*b = \hat{a}(\hat{x})b$ for all $a \in A$. Consider $\rho: A*b \to \mathbb{C}$ defined by $\rho(a*b) = \hat{a}(\hat{x})$ for all $a \in A$. Note that if $a_1 \in A_c$ with $\hat{a}_1 \equiv 1$ on a relatively compact nbhd. of $\hat{x}$, then $a_1*b = b$ and so $\rho(b) = \rho(a_1*b) = \hat{a}_1(\hat{x}) = 1$. For any $\alpha \in \mathbb{C}$, we then have $\rho(\alpha b) = \rho(\alpha(a_1*b)) = \rho((\alpha a_1)*b) = \alpha a_1(\hat{x}) = \alpha\hat{a}_1(\hat{x}) = \alpha$. It follows that ρ is well-defined, linear, and onto $\mathbb{C}$. Moreover, if $\rho(a*b) = 0$ then $\hat{a}(\hat{x}) = 0$ and since $I_{\hat{x}} = J(\{\hat{x}\})$ by <u>A6</u> we have $a*b = 0$. Thus, ρ is one-to-one. The relation $|\rho(a*b)| = |\hat{a}(\hat{x})\rho(b)| \leq ||\hat{a}||_\infty$ entails ρ is continuous on $A*b$. Extending ρ to $\overline{A*b} = [b]$ by continuity, we obtain a continuous linear functional from $[b]$ onto $\mathbb{C}$. Therefore, $\dim([b]) = 1$.

(iii)$\Longrightarrow$(ii) Suppose $\dim([b]) = 1$, then $[b] \simeq \mathbb{C}$ by an isomorphism h. We note that $(\mathbb{C},\hat{x})$, the space $\mathbb{C}$ with module multiplication $a*_{\mathbb{C}}\alpha = \hat{a}(\hat{x})\alpha$, $a \in A$, $\alpha \in \mathbb{C}$, is an essential Banach A-module. Denoting the image of b under h by $h(b)$, we have $h(b) \in \overline{A*_{\mathbb{C}}\mathbb{C}}$. This entails that for any $a \in A$, $h(a*b) = a*_{\mathbb{C}}h(b) = \hat{a}(\hat{x})h(b) = h(\hat{a}(\hat{x})b)$

using the fact that h is an A-module homomorphism. Since h is one-to-one, we have $a*b = \hat{a}(\hat{x})b$.

(ii) $\Longrightarrow$ (i) Suppose $\hat{y} \neq \hat{x}$, then there exists an $a \in A$ such that $\hat{a}(\hat{x}) = 0$ and $\hat{a}(\hat{y}) \neq 0$. Now (ii) implies $a*b = \hat{a}(\hat{x})b = 0$. Since $b \neq 0$, $a*b = 0$, and $\hat{a}(\hat{y}) \neq 0$ implies $\hat{y} \notin sp(b)$. We obtain $sp(b)^c \supset \Delta(A)\setminus\{x\}$. Consequently, $sp(b) \subseteq \{\hat{x}\}$.

(i) $\Longrightarrow$ (iv) Proposition 3.13 entails $sp([b]) = sp(b) = \{\hat{x}\}$. Writing $J_{\hat{x}}$ for $J(\{\hat{x}\})$ we have $[b] \subseteq J_{\hat{x}}^{\perp B}$. Then $[b]^{\perp A} \supseteq J_x^{\perp B \perp A} \supseteq J_{\hat{x}} = I_{\hat{x}}$ where the latter equality follows from A6. But $I_{\hat{x}}$ is a maximal closed ideal and so $[b]^{\perp A} = I_{\hat{x}}$ (i.e., $[b]^{\perp A} \neq A$ because $[b] \neq \{0\}$). Annihilating we have that $[b]^{\perp A \perp B} = J_{\hat{x}}^{\perp B}$. Condition HB2 applies to give $[b] = J_{\hat{x}}^{\perp B}$. The equivalence of (i) and (ii) entail (iv).

(iv) $\Longrightarrow$ (i) Suppose $[b] = \{b' \in B : a*b' = \hat{a}(\hat{x})b'$ for all $a \in A\}$. Now $sp(b) = sp([b]) = sp(J_{\hat{x}}^{\perp B}) = \{x\}$. Q.E.D.

Remarks 4.9

1. What is required in the proof of (i) $\Longleftrightarrow$ (iv) is the additional hypothesis that $\{x\}$ is an S-B set. This is guaranteed by HB2.

2. What the above characterization suggests is the manner in which one should perhaps view the possible decomposition of closed submodules into one-dimensional submodules (see III, §5, B).

We now consider some immediate consequences of Theorem 4.9.

Corollary 4.10. Let B be a Banach A-module. If $b \in B$ has finite spectrum $\{\hat{x}_1, \ldots, \hat{x}_n\}$, then $a*b = \sum_{j=1}^{n} \hat{a}(x_j)b_j$ for all $a \in A$ where $sp(b_j) = \{\hat{x}_j\}$, $j = 1, \ldots, n$.

Proof: For each $j = 1, 2, \ldots, n$, let $a_j \varepsilon A$ satisfy

(1) $\sigma(a_j) \cap \{\hat{x}_i\}_{i=j} = \phi$.

(2) $\hat{a}_j \equiv 1$ on a nbhd. U_j of $\hat{x}_j$ with $U_j \cap \left(\bigcup_{i=1}^{j-1} U_i\right) = \phi$.

(This is possible by the regularity of A). Set $b_j = a_j * b$, $j = 1, 2, \ldots, n$. For each $j \varepsilon \{1, 2, \ldots, n\}$, we have

$$sp(b_j) \subseteq \sigma(a_j) \cap sp(b) \subseteq \{\hat{x}_j\}.$$

But, if $a_j * b = 0$, then $\hat{a}_j(\hat{x}_j) = 0$, a contradiction, so that $sp(b_j) \neq \phi$. Hence, $sp(b_j) = \{\hat{x}_j\}$. By proposition 3.7 (ii), $\sum_{j=1}^{N} \hat{a}_j \equiv 1$ on a nbhd. of $sp(b)$ (namely $\bigcup_{j=1}^{n} U_j$) implies

$$b = (\sum_{j=1}^{n} a_j) * b = \sum_{j=1}^{n} (a_j * b) = \sum_{j=1}^{n} b_j .$$

Applying Theorem 4.9 to each b_j, for $a \varepsilon A$ we obtain

$$a*b = a*(\sum_{j=1}^{n} b_j) = \sum_{j=1}^{n} a*b_j = \sum_{j=1}^{n} \hat{a}(\hat{x}_j) b_j . \qquad \text{Q.E.D.}$$

Remark 4.10

We may apply proposition 3.8 to observe that the decomposition in Corollary 4.10 is unique.

For the classical case $B = C(T)$ and $L^1(T)$, Corollary 4.10 may be interpreted as follows:

Let $p(x) = \sum_{n=1}^{N} c_n e^{i\lambda_n x}$ be a trigonometric polynomial, then for

$f \in L^1(T)$, $f*p(x) = \sum_{n=1}^{N} \hat{f}(\lambda_n)c_n e^{i\lambda_n x}$ where $a = f$, $b = p$ and $b_j = c_i e^{i\lambda_j x}$. In particular, if $N = 1 = c_1$, and $\lambda_1 = n$, we have

$$f*e^{inx} = \hat{f}(n)e^{inx}.$$

This relation is crucial in showing that closed translation invariant subspaces of $C(T)$ are closed submodules of $C(T)$ as an $L^1(T)$-module. This is one instance in which the properties of elements with one point spectra play a vital role (see Edwards [37, §11]).

We now point out an alternate way of viewing the relation in the conclusion of Corollary 4.10. Our intention is not to purport Theorem 4.9 as an astounding discovery (for indeed part of it is known), but to stress the general relationship to significant works which have not been viewed in a module context. For example, Reiter proves as a lemma to an important result (theorem [90, p. 508]) the following:

> <u>Lemma (Reiter [90, p. 506])</u> If the hull E of a closed ideal $I \subseteq L^1(G)$ is countable and independent, then any $\phi \in I^{\perp\infty}$ is of the form
> $$\phi(x) = \sum_{\hat{x} \in E} a_{\hat{x}}(x, \hat{x})$$
> where $\sum_{x \in E} |a_{\hat{x}}| < \infty$.

Theorem 4.9 and its corollary lead us to the following finite version of Reiter's lemma.

<u>Proposition 4.11</u>. Let B be a Banach A-module and I a closed ideal with $\mathrm{hull}(I) = \{\hat{x}_j\}_{j=1}^{n}$. For each $b \in I^{\perp B}$, there exists a finite set $\{b_i\}_{i=1}^{n} \subset B$ satisfying $sp(b) \subseteq \{x_j\}_{j=1}^{n}$ and $a*b = \sum_{j=1}^{n} \hat{a}(\hat{x}_j)b_j$ for all $a \in A$.

Proof: Let $b \in I^{\perp B}$. Observe that $I \subseteq I^{\perp B \perp A} \subseteq b^{\perp A}$. From this we obtain $sp(b) = hull(b^{\perp A}) \subseteq hull(I) = \{\hat{x}\}_{j=1}^{n}$. The relation $a*b = \sum_{j=1}^{n} \hat{a}(\hat{x}_j)b_j$ for all $a \in A$ follows from Corollary 4.10. Q.E.D.

It is not difficult to obtain the conclusion in 4.11 for the hull(I) being any set of isolated points, but such a requirement would be severe. Our desire is to extend Reiter's lemma to modules but we have not been successful with our present methods. Observe that proposition 4.11 does not use the hypothesis of independence. Indeed, what one needs is to be able to carry out a construction analogous to that in Corollary 4.10 for hull(I). In particular, one must capture the essential properties of independence for the space $\Delta(A)$ which need not be a group. With respect to this, it is of interest to cite a result in [69, prop. 1.1] which we state for completeness.

Proposition 4.12. Let $\{M_{\hat{x}} : \hat{x} \in \Delta(A)\}$ be the family of τ-closed submodules where $M_{\hat{x}} = \{b \in B : a*b = \hat{a}(\hat{x})b$ for all $a \in A\}$ for each $\hat{x} \in \Delta(A)$. The family $\{M_{\hat{x}} : x \in \Delta(A)\}$ is linearly independent.

Before we proceed to the final result of this section, we pose the following question stemming from the preceding considerations.

Question: Is Reiter's lemma valid in the module context?

Our feeling is that the lemma is true. Reiter's proof involves techniques particular to the case in question, namely $A = L^1(G)$, $B = L^\infty(G)$. Reasons why we tend to believe the validity of the lemma for modules in this form will be more apparent in the next chapter. In particular, we cite Theorem 5.15 and Remark 5.17 (3).

As another application of Theorem 4.9, we obtain an alternate description of spectrum for a restricted class of modules.

<u>Theorem 4.13</u>. Let B be a Banach A-module such that $\Delta(A)$ is identifiable as a subset of B. Suppose B satisfies <u>HB2</u> for the τ-topology and b is a nonzero element in B, then

$$\hat{x} \in sp(b) \text{ if and only if } \hat{x} \in \overline{[b]}^{\tau}.$$

<u>Proof</u>: By Theorem 4.9, for each $a \in A$, $a*\hat{x} = \hat{a}(\hat{x})\hat{x}$. Now suppose $\hat{x} \in sp(b)$ and $a \in [b]^{\perp A}$. Then $\hat{a}(\hat{x}) = 0$, and so $0 = \hat{a}(\hat{x})\hat{x} = a*\hat{x}$ implies $\hat{x} \in [b]^{\perp A \perp B}$. By condition <u>HB2</u>, we have $\hat{x} \in [b]^{\perp A \perp B} = (\overline{[b]}^{\tau})^{\perp A \perp B} = \overline{[b]}^{\tau}$. For the reverse implication, let $\hat{x} \in \overline{[b]}^{\tau}$ and $a \in [b]^{\perp A}$. Since $[b]^{\perp A} = (\overline{[b]}^{\tau})^{\perp A}$ and $b \neq 0$, $0 = a*\hat{x} = \hat{a}(\hat{x})\hat{x}$ implies $\hat{a}(\hat{x}) = 0$. Therefore, $\hat{x} \in sp(b)$. Q.E.D.

<u>Remark 4.13</u>

We point out two special cases included in Theorem 4.13. For the weak-star topology on $B = L^{\infty}(G)$ as an $L^1(G)$-module, we have the usual definition of spectrum for elements in $L^{\infty}(G)$, that is, for $\varphi \in L^{\infty}(G)$, $\hat{x} \in sp(\varphi)$ if and only if $\hat{x}$ is in the weak-star closure of the translation invariant subspace generated by φ. The other case is for τ, the norm topology, and algebra modules B satisfying <u>HB2</u>. For $\varphi \in B$,

$$sp(\varphi) = \hat{G} \cap [\varphi] = \{\hat{x} \in G : \hat{x} \text{ is in the norm closure of } A*\varphi\}.$$

This is the <u>norm-spectrum</u> (see Katznelson [67, p. 159]).

We now proceed to a characterization of submodules whose spectrum have a particular structure. This embodies the ideas of §2-§3.

§4. Angular Semigroups and Submodules

We seek a characterization of closed submodules of algebra modules in terms of spectra. The spirit is that of de Leeuw and Mirkil [23].

Let us consider an example to illustrate our intention.

Example [23, p. 361]

Let M be a closed translation invariant subalgebra of $C_o(G)$. We assert that $0 \varepsilon \mathrm{sp}(M)$ if and only if M contains an approximate identity with respect to pointwise multiplication. To prove this assertion, we first observe that the $L^1(G)$-module $C_o(G)$ satisfies the hypothesis of Theorem 4.13 and so $0 \varepsilon \mathrm{sp}(M)$ if and only if $1 \varepsilon M$ (by [23, lemma 2.3] the uniform closure and weak-star closure with respect to $L^1(G)$ coincide).

Now suppose M has an approximate identity $\langle e_\alpha \rangle$ with respect to pointwise multiplication. The operator $E_\alpha : \varphi \to e_\alpha \cdot \varphi$ is bounded in the operator topology since for each $\varphi \varepsilon M$, $e_\alpha \cdot \varphi$ converges uniformly to φ. By the uniform boundedness principle, $||E_\alpha||$ is bounded. Since $||E_\alpha e_\alpha||_\infty = ||e_\alpha||_\infty ||e_\alpha||_\infty$, $||e_\alpha||_\infty$ is bounded. Let K be a compact subset of G. By translation invariance of M, for each $y \varepsilon K$, there exists a nbhd. V_y and some $\varphi_y \varepsilon M$ such that $\inf\{ |\varphi_y(x)| : x \varepsilon V_y\} > 0$. Since $||e_\alpha \cdot \varphi_y - \varphi_y||_\infty = 0$, $\langle e_\alpha \rangle$ converges uniformly to 1 on V_y. Compactness of K insures $\langle e_\alpha \rangle$ converges uniformly to 1 on K. Taking $||e_\alpha||_\infty = 1$ and noting that $|e_\alpha(x) - 1| \to 0$ uniformly on K, we see that $e_\alpha \to 1$ in the "narrow" topology. Hence, $e_\alpha \to 1$ in the weak-star topology. That is $1 \varepsilon \overline{M}^{w^*}$ (weak-star closure), and $0 \varepsilon \mathrm{sp}(M)$. For the reverse implication let $0 \varepsilon \mathrm{sp}(M)$. Then $1 \varepsilon \overline{M}^{w^*}$ and so there exists a net $\langle e_\alpha \rangle \subset M$ such that $\langle e_\alpha \rangle$ weak-star converges to 1. Thus, $e_\alpha \to 1$ narrowly (see [23, lemma 6.1]) and so $||e_\alpha||_\infty \to 1$ and $|e_\alpha(x) - 1| \to 0$ uniformly on compact subsets of G. For $\varphi \in C_o(G)$, say $||\varphi||_\infty = 1$, let K be a compact subset of G so that for $\varepsilon > 0$, $|e_\alpha(x) - 1| < \varepsilon$ and $|\varphi(x)| < \varepsilon/2$ if $x \notin K$. Then

$$|e_\alpha(x)\varphi(x) - \varphi(x)| \leq |e_\alpha(x) - 1||\varphi(x)| < \varepsilon \quad \text{for} \quad x \in G \quad \text{and}$$

hence, $e_\alpha\varphi - \varphi \to 0$ uniformly on G.

Now for a characterization of submodules having the above "approximate identity" property.

<u>Proposition 4.14</u>. Let B be an algebra A-module. Suppose M is a closed submodule of B and there exists a subalgebra $A_o \subset A$ such that $A_o \cap M$ is dense in M, then $sp(M) = c\ell(int(sp(M)))$.

<u>Proof</u>: Let $x \in sp(M)$ (the conclusion being trivial for $M = \{0\}$). Now there exists a net $\langle \hat{x}_\alpha \rangle \subset \bigcup_{\varphi \in M} sp(\varphi)$ such that $\langle \hat{x}_\alpha \rangle$ converges to $\hat{x}$.

Fix α and consider $\hat{y} = \hat{x}_\alpha$. There is an element $\varphi \in M$ such that $\hat{y} \in sp(\varphi)$. Since $A_o \cap M$ is dense in M, $||\varphi_\beta - \varphi||_B \to 0$ for some net $\langle \varphi_\beta \rangle \subset A_o \cap M$. For each nbhd. V of $\hat{y}$, there is some β_V satisfying $sp(\varphi_{\beta_V}) \cap V \neq \phi$. But we have $\varphi_\beta \in A_o$ entailing $sp(\varphi_\beta) = \sigma(\varphi_\beta) = c\ell\{\hat{x} \in \Delta(A) : \hat{\varphi}_\beta(\hat{x}) \neq 0\}$. For each nbhd. V of $\hat{y}$, let $\hat{y}_V \in \{\hat{x} \in \Delta(A) : \hat{\varphi}_{\beta_V}(x) \neq 0\} \cap V$ (use the regularity of A and the fact that $Z(\varphi_\beta)^c$ is open). We assert that $\{\hat{y}_V : V$ is a nbhd. of $\hat{y}\}$ converges to $\hat{y}$. To see this, let W be any nbhd. of $\hat{y}$. By regularity, there is a nbhd. of $\hat{y}$ such that $\overline{V} \subset W$. Then for W there is a $\hat{y}_W$ and for V a $\hat{y}_V$ satisfying $\hat{y}_W \in W$ and

$\hat{y}_V \in V \subset \overline{V} \subset W$. Therefore, for $V \subset W$, $y_V \in W$. Then $\langle \hat{y}_V \rangle$ is eventually in W. We have $\hat{y}_V \to \hat{y}$. But then $\hat{y}_V \in \mathrm{int}(\mathrm{sp}(\varphi_{\beta_V})) \subset \mathrm{int}(\mathrm{sp}(M))$ implies $\hat{y} \in c\ell\ \mathrm{int}(\mathrm{sp}(M))$. Since $\hat{y} = \hat{x}_\alpha$ is an arbitrary element in $\langle \hat{x}_\alpha \rangle$, $\hat{x}_\alpha \to \hat{x}$ entails $\hat{x} \in c\ell\ \mathrm{int}(\mathrm{sp}(M))$. Hence, $\mathrm{sp}(M) = c\ell(\mathrm{int}(\mathrm{sp}(M)))$.

Q.E.D.

Theorem 4.15. Let B be a Banach algebra A-module satisfying HB2. Suppose M is a closed submodule of B with $\mathrm{sp}(M)$ an S-B set. The set $B_o \overset{\mathrm{def}}{\equiv} \{\varphi \in B : \mathrm{sp}(\varphi) \subset \mathrm{int}(\mathrm{sp}(M)),\ \mathrm{sp}(\varphi) \text{ compact}\}$ is a dense submodule of M.

Proof: For $\varphi \in B_o$, $\mathrm{sp}(\varphi) \subset \mathrm{int}(\mathrm{sp}(M))$ together with the fact that $\mathrm{sp}(M)$ is an S-B set imply $\varphi \in M$. Therefore, $B_o \subseteq M$. For φ_1, $\varphi_2 \in B_o$, $\mathrm{sp}(\varphi_1 + \varphi_2) \subseteq \mathrm{sp}(\varphi_1) \cup \mathrm{sp}(\varphi_2) \subset \mathrm{int}(\mathrm{sp}(M))$ and so $\mathrm{sp}(\varphi_1 + \varphi_2)$ being compact as well entails $\varphi_1 + \varphi_2 \in B_o$. Evidently, B_o is a subspace of M. Furthermore, for $f \in A$, $\mathrm{sp}(f*\varphi) \subseteq \mathrm{sp}(\varphi) \subset \mathrm{int}(\mathrm{sp}(M))$ for $\varphi \in M$, hence $f*\varphi \in B_o$. So B_o is a submodule of M.

Suppose there is a $g \in B_o^{\perp}$ with $g \notin M^{\perp}$. Then $\mathrm{sp}(B_o) = \mathrm{hull}(B_o^{\perp}) \neq \mathrm{hull}(M^{\perp}) = \mathrm{sp}(M)$ by spectral synthesis. Now there exists an $\hat{x}_o \in \mathrm{sp}(M)$ such that $\hat{h}(\hat{x}_o) \neq 0$ for some $h \in B_o^{\perp}$. Let $f \in A$ be such that $\hat{f}(\hat{x}_o) \neq 0$ and $\sigma(f) \subset \mathrm{int}(\mathrm{sp}(M))$ with $\sigma(f)$ compact. For $\varphi \in M$, $f*\varphi \in B_o$ so that $h*f*\varphi = 0$. This implies $h*f \in M^{\perp}$ and hence $\widehat{h*f}(\hat{x}_o) = 0$. But then $\hat{h}(\hat{x}_o) \neq 0$ and $\hat{f}(\hat{x}_o) \neq 0$ render a contradiction. Thus, $B_o^{\perp} = M^{\perp}$ and condition HB2 entails $M = c\ell(B_o)$.

Q.E.D.

Lemma 4.16. Let B be a Banach $L^1(G)$-module satisfying HB2. If E is a closed angular subsemigroup of $\hat{G}$, then E is an S-B set.

Proof: By I, §4, Fact 12, E is an $S\text{-}L^1(G)$ set. By Theorem 4.1 (ii)$\Leftrightarrow$(iv), HB2 guarantees that E is an $S\text{-}B$ set. Q.E.D.

Remark 4.16

This directly extends lemma 6.3 in [23]. The proof in [23] involves techniques particular to the case $B = C_o(G)$, but we have freed ourselves of this by application of Theorem 4.11.

Corollary 4.17. Let B be a Banach $L^1(G)$-module satisfying HB2. Let M be a closed submodule of B with $sp(M)$ angular, then B_o is a dense submodule of M.

Proof: This is merely an application of Theorem 4.15 and lemma 4.16 since angular semigroups are of synthesis. Q.E.D.

Remark 4.17

If, in addition, B is an algebra (with respect to convolution) then B_o is easily verified to be a subalgebra. That is, for φ_1, $\varphi_2 \in B_o$, $sp(\varphi_1 * \varphi_2) \subseteq sp(\varphi_1) \cap sp(\varphi_2) \subset int(sp(M))$ so that $\varphi_1 * \varphi_2 \in M$.

In the event that we deal with algebra modules which are themselves algebras, we can characterize certain submodules with respect to the structure of their spectrum.

Theorem 4.18. Let B be a Banach $L^1(G)$-module which is also an algebra. Suppose B satisfies HB2 and that M is a closed submodule which is also a subalgebra. Then the following are equivalent:

(i) $sp(M)$ is angular

(ii) $1 \in M$ and B_o is a dense submodule of M which is a subalgebra.

<u>Proof</u>: (i) $\Longrightarrow$ (ii) Applying theorem 4.13, $0 \in sp(M)$ entails $1 \in M$. That B_o is a dense submodule and subalgebra of M is the content of Corollary 4.17 and Remark 4.17.

(ii) $\Longrightarrow$ (i) Again appealing to 4.13, $1 \in M$ entails $0 \in sp(M)$. The fact that $sp(M)$ is a closed semigroup follows from theorem 4.7. To show that $sp(M)$ is angular, we only need verify that the hypothesis of proposition 4.14 are satisfied to obtain $sp(M) = c\ell(int(sp(M)))$. But (ii) implies $B_o = B_o \cap M$ is dense in M, and scrutiny of the proof of proposition 4.14 reveals that we may take $A_o = B_o$ without A_o contained in A (i.e., we only need that A_o is an algebra). Q.E.D.

<u>Remark 4.18</u>

For $B = C_o(G)$, $A = L^1(G)$ we arrive at the result in [23], which shows there is a one-to-one correspondence between the closed translation invariant subalgebras of $C_o(G)$ which have a pointwise approximate identity and contain dense subalgebras with the angular subsemigroups of $\hat{G}$.

§5. Closure Properties and Spectra

In §5 of Chapter III we defined the closure and decomposition properties for Banach modules. Our purpose in this section is to present conditions which are sufficient for the closure property to obtain. Equivalence of the closure and decomposition properties subsists for a class of modules. As remarked earlier, we follow Reiter in spirit, but maintain a persistency in demonstrating the suitability of working in modules. Our main contribution is one of recognition and utilization of the properties of spectra in a "proper" context.

Our first result concerns an analogue of a theorem for Banach algebras concerning the representation of ideals in case their hull is a disjoint union of closed sets. We state this result without proof mentioning that <u>A1-A3</u> and local membership (recall I, §2) are crucial in the proof.

<u>Proposition 4.19</u>. Let A be a commutative Banach algebra and I be a closed ideal of A with $\text{hull}(I) = E_1 \cup E_2$ where E_1 and E_2 are non-empty disjoint closed subsets of $\Delta(A)$. Then I can be uniquely expressed in the form $I = I_1 \cap I_2$ where I_1 and I_2 are closed ideals of A with $\text{hull}(I_i) = E_i$, $i = 1, 2$.

In fact, $I_i = c\ell(I + J_i)$ where J_i $J(E_i)$, $i = 1, 2$.

<u>Proof</u> [91, pp. 557-559]. We note that it is possible to derive the following analogue of proposition 4.19 by duality if we add the condition of τ-bi-annihilation invariance. However, invoking a decomposition result previously obtained for elements, namely proposition 3.8, we are able to derive it for arbitrary essential Banach modules without the full machinery mentioned for the proof of 4.19.

<u>Theorem 4.20</u>. Let B be an essential Banach A-module and M a τ-closed submodule of B with $sp(M) = E_1 \cup E_2$ where E_1 and E_2 are non-empty disjoint closed subsets of $\Delta(A)$. Then M can be expressed in the form $M = c\ell_B^\tau(M_1 + M_2)$ where $M_i = M \cap N_i$, $N_i \equiv J(E_i)^{\perp B}$, $i = 1, 2$.

<u>Proof</u>: Let M_i be as in the statement of the theorem. We clearly have $M_1 + M_2 \subseteq M$. Suppose $b \varepsilon M$ and $sp(b)$ is compact. Then if $F_i = sp(b) \cap E_i$, $i = 1, 2$, we have F_1 and F_2 a pair of disjoint, compact subsets of $\Delta(A)$ with $sp(b) = F_1 \cup F_2$. By proposition 3.8, there is a unique decomposition $b = b_1 + b_2$ where $sp(b_i) \subseteq F_i$, $i =$

1, 2. Hence, $b_i \in M_i$, $i = 1, 2$ and we see that $b \in M_1 + M_2$. Thus, if $B_c = \{b \in B : sp(b) \text{ is compact}\}$, $B_c \cap M \subseteq M_1 + M_2$. By Theorem 3.9, B_c is dense in B and hence B_c is τ-dense in B. Since $B_c = A_c * B$, one obtains $A_c * M = (A_c * B) \cap M = B_c \cap M \subseteq M_1 + M_2 \subseteq M$. Thus, A2 entails (with the fact that B is essential) that $\overline{A_c * M} = M$ and consequently, $M = c\ell(M_1 + M_2)$. In particular $M = c\ell_B^\tau(M_1 + M_2)$. Q.E.D.

Our next objective is to determine conditions under which the decomposition and closure properties are equivalent for a pair (E_1, E_2) of non-empty disjoint sets. We proceed with a lemma which appears crucial for such an equivalence and confines the class of modules for which Theorem 4.22 applies.

Lemma 4.21. Let $B = A^*$ be regarded as a Banach A-module. Suppose I and J are closed ideals in A, then $I + J$ is closed if and only if $I^{\perp B} + J^{\perp B}$ is closed.

Proof: The proof in [91] carries over for dual modules and we reproduce the essential idea for completeness. Let $S : A \to A/I$ be the canonical map. Now $I + J$ is closed in A if and only if $S(J)$ is closed in A/I. Let T be the restriction of S to J. Then T is a bounded linear map of $J \to A/I$. Thus, the adjoint $T^* : I^{\perp B} \to A^*/J^{\perp B}$ is bounded and linear (note that lemma 3.20 applies to show $I^{\perp B} = I_\perp = (A/I)^*$, the latter a property of dual annihilators [35]). But the range of T is closed if and only if the range of T^* is closed in A^*. The conclusion follows since $T(J) = I + J$ and $T^*(I^{\perp B}) = I^{\perp B} + J^{\perp B}$.

Q.E.D.

Theorem 4.22. Let $B = A^*$ be regarded as a Banach A-module. Suppose E_1 and E_2 are non-empty disjoint subsets of $\Delta(A)$, then (E_1, E_2) has the closure property with respect to A if and only if it has the decomposition property with respect to B.

Proof: Suppose (E_1, E_2) has the closure property with respect to A. Let $b \in B$ be any element whose spectrum is of the form $sp(b) = F_1 \cup F_2$ and consider $J(F)$. We assert that $J(F) = J(F_1) \cap J(F_2)$ (where as usual $J(E)$ is the smallest closed ideal with hull E). To see this, we apply proposition 4.19 with $I = J(F)$. Then $J(F) = J_1 \cap J_2$ uniquely where J_i are closed ideals with hull F_i, $i = 1, 2$. Observe that $J_i = c\ell(J(F) + J(F_i))$ by 4.19 and the assertion follows. By the closure property, $J(F_1) + J(F_2) = A$ and so $J(F_1) + J(F_2)$ is closed. Applying proposition 3.23 (note B is weak-star bi-annihilation invariant) and lemma 4.21 we obtain

$$J(F)^{\perp B} = (J(F_1) \cap J(F_2)^{\perp B}) = c\ell(J(F_1)^{\perp B} + J(F_2)^{\perp B}) = J(F_1)^{\perp B} + J(F_2)^{\perp B}.$$

Thus, $b = b_1 + b_2$ where $sp(b_i) = F_i$, $i = 1, 2$, and so (E_1, E_2) has the decomposition property with respect to B.

For the other implication, suppose (E_1, E_2) has the decomposition property with respect to B. Let I_1 and I_2 be closed ideals in A with $\mathrm{hull}(I_i) \subseteq E_i$, $i = 1, 2$. The sets $F_i = \mathrm{hull}(I_i)$, $i = 1, 2$, are closed, and we may assume non-empty since otherwise Wiener's theorem applies (I, §3, Fact 4). Now set

$$J(F)^{\perp B} = \{b \in B : sp(b) \subseteq F \equiv F_1 \cup F_2\}.$$

Now (F_1, F_2) also has the decomposition property and hence $J(F)^{\perp B} = J(F_1)^{\perp B} + J(F_2)^{\perp B}$. But $J(F_1)^{\perp B} + J(F_2)^{\perp B}$ is then closed in B and lemma 4.21 entails $J(F_1) + J(F_2)$ is closed in A. Therefore, $J(F_1) + J(F_2) = A$ since $\mathrm{hull}(J(F_1) + J(F_2)) = \emptyset$. But $J(F_i) \subseteq I_i$,

$i = 1, 2$ implies $I_1 + I_2 = A$. Therefore, (E_1, E_2) has the closure property with respect to A. Q.E.D.

Remark 4.22

We have used the duality between A and B in Theorem 4.22 in a very essential way. Seemingly, τ-bi-annihilation invariance could be imposed to obtain the equivalence of the closure and decomposition properties in such modules, but we have not been successful in our attempts.

Additional conditions have been given by Reiter [91] to guarantee the decomposition and closure properties in the case $L^1(G)$, $L^\infty(G)$. The techniques involved are particular to $L^1(G)$. One may possibly extend to algebra modules such conditions, but this may take us astray from our theme. That is, we have exhibited that the decomposition and closure properties may be investigated in a module setting, which reaffirms a continuity of regarding spectral synthesis and spectral analysis problems in Banach modules.

With this in mind, we close this section with a brief indication of such a program--but only for Segal and Beurling algebras. We first state a proposition due to Reiter for Segal algebras [93] and Kerlin for Beurling algebras [68]. Recall that G is a LCAG.

Proposition 4.23. Let A be a Segal (Beurling) algebra. Let H be a closed subgroup of G and $A(G/H)$ the image of $A(G)$ under the mapping $T_H : L^1(G) \to L^1(G/H)$ defined by $T_H f(\dot{x}) = \int_H f(x + s)ds$ $\dot{x} \in G/H$, $\dot{x} = \Pi_H(x)$ where Π_H is the canonical map. Then the following holds:

(i) $A(G/H)$ is a Segal (Beurling) algebra with the quotient norm:

$$A(^G/H) \cong {}^{A(G)}/_{J_A(G,\ H)}$$

where $J_A(G, H)$ is the kernel of the restriction T_H to $A(G)$;

(ii) the image under T_H of a closed ideal of $A(G)$ is a closed ideal of $A(G/H)$.

Proof: (We refer the reader to the citations preceding the statement of the theorem. The proof is omitted as it is not pertinent to the theme of this thesis.)

Theorem 4.24. Let A be a Segal (Beurling) algebra. Let (E_1, E_2) be a pair of non-empty disjoint subsets of $\hat{G}$. If Γ is a closed subgroup of $\hat{G}$ with $E_1 \subset \Gamma$ and $E_2 \cap \Gamma = \phi$, then (E_1, E_2) has the closure property with respect to A.

Proof: Let Γ be a closed subgroup of $\hat{G}$ satisfying $E_1 \subset \Gamma$ and $E_2 \cap \Gamma = \phi$. Since Γ is a closed subgroup, Γ is of synthesis and so $I(\Gamma) = J(\Gamma)$. Let $H = \hat{\Gamma}$, then $I(\Gamma) = J_A(G, H)$. For any closed ideal I with $\text{hull}(I) \subset \Gamma$, $I(\Gamma) \subset I$ and so it suffices to show that for any closed ideal I_1 with $\text{hull}(I_1) \cap \Gamma = \phi$, $I(\Gamma) + I_1 = A$. But proposition 4.23 applies to show $I(\Gamma) + I_1$ is a closed ideal for any closed ideal I_1 with $\text{hull}(I_1) \cap \Gamma = \phi$. Hence, $\text{hull}(I(\Gamma) + I_1) = \Gamma \cap \text{hull}(I_1) = \phi$. By Wiener's Theorem (I, §3, Fact 4), $I(\Gamma) + I_1 = A$ and so (E_1, E_2) has the closure property. Q.E.D.

CHAPTER V

A WIENER-DITKIN CONDITION AND ALMOST PERIODICITY

The difficulty of the general problem of spectral synthesis is well-appreciated, however, there are instances in which sets of spectral synthesis can be determined. It is well known that in any regular semisimple Banach algebra A satisfying "Ditkin's condition," a closed subset E of $\Delta(A)$ is a set of spectral synthesis for A if the boundary of E contains no nonempty perfect subsets (I, §3, Fact 8). This is a consequence of the Wiener-Ditkin-Shilov Theorem which provides the "best" sufficient conditions that are known for closed subsets of $\Delta(A)$ to be of spectral synthesis. If our approach to spectral synthesis (analysis) is to be a "good way" of looking at the general problem, then a Wiener-Ditkin-Shilov Theorem should exist in our module context. The first part of this chapter is devoted to determining that in a "weak sense" this is, indeed, the case. The development is analogous to the standard treatment in the theory for Banach algebras. The second part of the chapter reveals applications of spectra in modules to almost periodicity. Our primary contributions are characterizations of almost periodic elements in the spirit of Loomis and a module formulation of a theorem of Beurling as well.

§1. Basic Concepts: Condition (D) and Local Membership

Our first concern is to formulate a definition of cospectrum for elements in Banach modules. As is customary, A will denote a commutative Banach algebra and B a Banach A-module satisfying the basic assumptions A1-A6.

The problem of obtaining a "zero set" for an element $b \in B$ appears to be very difficult. We intend to utilize what is perhaps the "zero set" which is most easily recaptured from the spectrum of b. This will lead to a weaker form of condition (D) and, consequently, a weaker form of Wiener-Ditkin-Shilov Theorem. Nevertheless, our intention is to demonstrate that sufficient conditions for spectral synthesis are accessible in Banach modules in a manner analogous to that used in Banach algebras. Moreover, if a "better" definition of cospectrum is made available, there would be a sharpening of some of the results to follow. For the present, we must be satisfied since the theme of this thesis is again sustained.

We proceed to our definition of cospectrum.

Notation: For $b \in B$, $\Sigma(b) \overset{\text{def}}{\equiv} sp(b) \setminus \partial sp(b)$

Observe that $\Sigma(b) \equiv int(sp(b))$ and hence may at times be empty, for example in the case if b has one-point spectrum.

Definition 5.1. Let $b \in B$ and M be a submodule of B. The p-cospectrum of b relative to B is defined in this way

$$p\text{-}cosp(b) \equiv \Sigma(b)^c;$$

and the p-cospectrum of M relative to B is defined in this way

$$\text{p-cosp}(M) \equiv \bigcap_{m\varepsilon M} \text{p-cosp}(m).$$

Remarks 5.1

1. Since $\Sigma(b)$ is open, both p-cosp(b) and p-cosp(M) are closed.

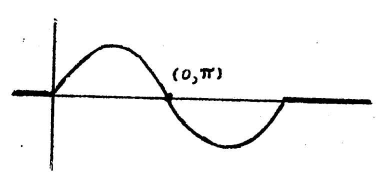
2. The term cospectrum is due to L. Schwartz [97] for the zero set of the Gelfand transform of an element in A. We also refer to Reiter [92] (cf. I, §2).

3. Writing Z(a) for the zero set of $\hat{a}$, $a \varepsilon A$, the usual definition of cospectrum of $a \varepsilon A$ is given by the equality cospectrum(a) = cosp(a) = Z(a). It is clear that if $b \varepsilon A \cap B$, then p-cosp(b) $\subseteq$ Z(b).

For $a \varepsilon A$, it is certainly possible that int(sp(a)) $\cap$ Z(a) be nonempty. In general, p-cosp(b) will fail to include all the points in sp(b) $\cap$ Z(b), where $b \varepsilon A \cap B$, as the following example illustrates.

Example:

$$\text{Set} \quad \Psi(x) = \begin{cases} \sin(x) & 0 \le x \le 2\Pi \\ 0 & \text{elsewhere} \end{cases}$$



Now $\Psi \varepsilon C_o(\mathrm{IR})$ satisfies $Z(\Psi) = (-\infty,0] \cup \{\Pi\} \cup [2\Pi,\infty)$ and $\Sigma(\Psi) = (0,2\Pi)$. Thus, $\Sigma(\Psi)^c = (-\infty,0] \cup [2\Pi,\infty)$ so that p-cosp(Ψ) =

$Z(\psi)\setminus\{\Pi\}$. The heart of the issue here comes from the fact that the <u>closed</u> support $\sigma(\psi)$ of a function ψ may contain zeros of ψ which are interior points of $\sigma(\psi)$. In case $\Delta(A)$ is discrete, then, indeed, we have $\text{p-cosp}(b) = Z(b)$ for $b \in A \cap B$.

In the event $\text{spt}(b) \overset{\text{def}}{\equiv} \{\hat{x} | \hat{b}(x) \neq 0\}$ for $b \in A \cap B$ is an open interval, say in the case $\Delta(A) = \mathbb{R}$, we can be more precise. Let us confine ourselves for the remainder of this paragraph to the case: B is a Banach A-module of functions defined on $\mathbb{R}$ and $\Delta(A) = \mathbb{R}$. Suppose $b \in A \cap B$ satisfies

$$Z(b) = \text{p-cosp}(b) \cup H(b)$$

for some $H(b) \subset \mathbb{R}$ which partitions $\text{spt}(b)$ into a union of disjoint open intervals (i.e., $\text{int}(\text{sp}(b))\setminus H(b) = \bigcup_{\alpha\in\Lambda} O_\alpha$, $O_\alpha \cap O_\beta = \phi$ for $\alpha = \beta$, and O_α is an open interval for each $\alpha\in\Lambda$), and such that $H(b) = \{x_{\alpha\beta} : \overline{O}_\alpha \cap \overline{O}_\beta = \{x_{\alpha\beta}\}$ for some $\alpha,\beta \in \Lambda\}$. Observe that $\text{sp}(b) = \bigcup_{\alpha\in\Lambda} \overline{O}_\alpha$, and that the points of $H(b)$ are isolated (there will always exist some set $H(b) \subset \Delta(A)$ satisfying $Z(b) = \text{p-cosp}(b) \cup H(b)$ for $b \in A \cap B$ in an arbitrary Banach A-module). In the case under consideration, one can explicitly express the p-cospectrum so that it coincides with the zero set. For each $\alpha\in\Lambda$, let b_α be the restriction of b to O_α on O_α and zero elsewhere. Then $b = \sum_{\alpha\in\Lambda} b_\alpha$. Since $\text{spt}(b_\alpha) = O_\alpha$, we also have $Z(b_\alpha) = O_\alpha^c = \text{p-cosp}(b_\alpha)$, and the fact that $\text{sp}(b)$ is closed entails $\bigcup_{\alpha\in\Lambda} \overline{O}_\alpha = \overline{\bigcup_{\alpha\in\Lambda} O_\alpha}$. Therefore,

$$\text{p-cosp}(b) = [sp(b)\setminus\partial sp(b)]^c = [\bigcup_{\alpha\varepsilon\Lambda}\overline{O}_\alpha\setminus\partial(\bigcup_{\alpha\varepsilon\Lambda}\overline{O}_\alpha)]^c = [\overline{\bigcup_{\alpha\varepsilon\Lambda} O_\alpha}\setminus(\overline{\bigcup_{\alpha\varepsilon\Lambda} O_\alpha})]^c$$

$$= (\bigcup_{\alpha\varepsilon\Lambda} O_\alpha)^c = Z(b).$$

The partitioning of b by $H(b)$ into the sum $\sum_{\alpha\varepsilon\Lambda} b_\alpha$ is crucial. An alternative for a more general situation would be to consider a "partition of unity," but to expect that the members of such a partition of unity (or the sum) belong to the proper space (i.e., the Banach module) may seem a little too ambitious at this point. Further remarks will be made in Chapter VI, in particular, we will indicate other "candidates" for cospectrum in essential modules. Although we have but defined a "partial" cospectrum, its adequacy will be apparent in subsequent work. In particular, we will draw on the relationship between p-cospectrum and spectrum.

With the above definition of p-cospectrum we formulate a Wiener-Ditkin condition for modules. This is notably weaker due to our use of p-cospectrum.

<u>Definition 5.2</u>. A Banach A-module B <u>p-satisfies condition (D) at $\hat{x} \varepsilon \Delta(A)$</u> if for each $b \varepsilon B$ with $\hat{x} \varepsilon$ p-cosp(b), there is a sequence $\langle a_n \rangle \subset A$ satisfying:

(i) $\hat{a}_n \equiv 0$ on U_n, a nbhd. of $\hat{x}$, for each n, and

(ii) $||a_n * b - b||_B \to 0$ as $n \to \infty$.

We say B satisfies condition (D) at infinity (for $\Delta(A)$ non-compact) if for each $b \varepsilon B$ there is a sequence $\langle a_n \rangle \subset A_c$ where $||a_n * b - b||_B \to 0$. If B p-satisfies condition (D) at all points of $\Delta(A)$ and at infinity, we say B <u>p-satisfies condition (D)</u>.

If the commutative Banach algebra A satisfies condition (D) as an algebra, then clearly A regarded as an A-module p-satisfies condition (D) as specified by definition 5.2. The containment p-cosp(b) $\subseteq$ Z(a) for $b \in A \cap B$ and the example above showing that it is, in general, proper indicates that the module version of condition (D) may not imply the algebra version, hence is in this sense "weaker." In particular, even though the algebra A may p-satisfy condition (D) as an A-module, the property for some point in a set $H(b)$ may not be obtained if the set $H(b)$ is not "well-behaved." However, by previous observations, p-cosp(b) = Z(b) for $A \cap B$ if $\Delta(A)$ is discrete and we can show that the concept of condition (D) in Banach algebras coincides with condition (D) in modules.

We next formulate an indispensable concept: local membership. The notion originated in the work of Wiener [109, p. 245] and subsequently developed for the group algebra and Banach algebras (cf., I, §3).

Definition 5.3. Let M be a submodule of a Banach A-module B. An element $b \in B$ belongs locally to M at $\hat{x} \in \Delta(A)$ if there exists a neighborhood U of $\hat{x}$ and an element $a \in A$ satisfying:

$$\hat{a} \equiv 1 \text{ on } U \text{ and } a*b \in M.$$

If $\Delta(A)$ is non-compact, b belongs locally to M at infinity if either of the following conditions hold:

(i) sp(b) is compact, or

(ii) there is an $a \in A$ with $a*b \in M$ such that $\hat{a} \equiv 1$ outside some compact set.

If b belongs locally to M at each point of $\Delta(A)$ and at infinity (if $\Delta(A)$ is non-compact), we say b belongs locally to M everywhere.

We now verify that our concept of "local membership" extends the usual notion as defined for Banach algebras (I, §3).

<u>Proposition 5.1</u>. Let A be a regular semisimple commutative Banach algebra. Let I be a closed ideal of A. An element $a \varepsilon A$ belongs locally to the ideal I at $\hat{x} \varepsilon \Delta(A)$ if and only if a belongs locally to the submodule I of A at $\hat{x} \varepsilon \Delta(A)$.

<u>Proof</u>: Suppose $a \varepsilon A$ belongs locally to I at $\hat{x}$ as a submodule. Now there is a nbhd. U of $\hat{x}$ and an $a_1 \varepsilon A$ such that $\hat{a}_1 \equiv 1$ on U and $aa_1 \varepsilon I$. But then $\widehat{a_1 a} = \hat{a}_1\hat{a} = \hat{a}$ on U and since $a_1 a \varepsilon I$, a belongs locally to I at $\hat{x}$ as an ideal.

For the converse, suppose $a \varepsilon A$ belongs locally to I at $\hat{x}$ as an ideal. Then there is an $a_o \varepsilon I$ such that $\hat{a} = \hat{a}_o$ on some nbhd. U of $\hat{x}$. Let V be a relatively compact nbhd. of $\hat{x}$ with $\overline{V} \subset U$. By the regularity of A, there is an $a_1 \varepsilon A$ such that $\hat{a}_1 \equiv 1$ on V and $\sigma(a_1) \subset U$. For $\hat{y} \notin U$, $\hat{a}_1(\hat{y}) = 0$ and so $\widehat{a_1 a_o}(\hat{y}) = a_1 a(\hat{y})$. If $\hat{y} \varepsilon U$, then $\hat{a}_o(\hat{y}) = \hat{a}(\hat{y})$ and hence $\hat{a}_1(\hat{y})\hat{a}_o(\hat{y}) = \hat{a}_1(\hat{y})\hat{a}(\hat{y})$ on U. We obtain $\widehat{a_1 a_o} = \widehat{a_1 a}$ on $\Delta(A)$. By semisimplicity, $a_1 a_o = a_1 a_o$. Since $a_1 a \varepsilon I$ and $\hat{a}_1 \equiv 1$ on V, we see that a belongs locally to I at $\hat{x}$ as a submodule.

Q.E.D.

For the remainder of this chapter, we make the following

<u>Assumption</u>: The Banach algebra A is self-adjoint. That is, for each $a \varepsilon A$, there is an $a_1 \varepsilon A$ such that $\hat{a}_1 = \overline{\hat{a}}$ (the complex conjugate of $\hat{a}$).

<u>Remarks 5.2</u>

1. The "a" in definition 5.3 may be taken to be in A_c by assumption <u>A2</u>.

2. The self-adjointness assumption is satisfied by the group algebra. Thus, group algebra modules remain as an important source of examples.

3. The above assumption entails that $\hat{A}$ is dense in $C(\Delta(A))$, see Loomis [81, 26B].

4. Most importantly, the assumption entails that if E is a closed subset of $\Delta(A)$ and $\hat{x} \notin E$, then there is an $a \in A$ such that $\hat{a} \equiv 1$ on a nbhd. of $\hat{x}$, $\hat{a} \equiv 0$ on E <u>and</u> $\hat{a} \geq 0$.

5. By remark (4), the element a in the definition of local membership can be assumed to satisfy $\hat{a} \geq 0$.

§2. Criteria for Local Membership

Our next result provides an essential tool in subsequent considerations. As in the standard result for Banach algebras (see for example Loomis [81, 25]), we use the concept of partition of unity. Recall $B_e \equiv$ essential part of B.

<u>Theorem 5.2</u>. Let B be a Banach A-module. If $b \in B_e$ belongs locally to a closed submodule M everywhere, then $b \in M$.

<u>Proof</u>: Let $\langle a_\alpha \rangle \subset A$ be such that $||a_\alpha * b - b||_B \to 0$ and $\sigma(a_\alpha)$ is compact. Then $sp(a_\alpha * b)$ is compact and $a_\alpha * b$ belongs locally to M at each point of $\hat{x} \in \Delta(A)$. To see this, let $\hat{x} \in \Delta(A)$; now there is an $a \in A$ and a nbhd. U of $\hat{x}$ such that $\hat{a} \equiv 1$ on U and $a*b \in M$. Thus, $a*a_\alpha*b \in M$ and $a_\alpha*b$ belongs locally to M at $\hat{x}$. It clearly suffices to prove $a_\alpha*b \in M$ since M is closed. For fixed (but arbitrary) α, let $b_o = a_\alpha*b$.

For each $\hat{x} \in sp(b_o)$, there is a relatively compact nbhd. $U_{\hat{x}}$,

$m_{\hat{x}} \varepsilon M$ and an $a_{\hat{x}} \varepsilon A$ satisfying $\hat{a}_{\hat{x}} \equiv 1$ on $U_{\hat{x}}$, $\hat{a}_{\hat{x}} \geq 0$ and $a_{\hat{x}}*b = m_{\hat{x}}$. Clearly, $\{U_{\hat{x}} : x \varepsilon \Delta(A)\}$ covers $sp(b_o)$. The compactness of $sp(b_o)$ entails that there are a finite number of elements $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n \varepsilon sp(b_o)$ such that $\{U_{\hat{x}_i}\}_{i=1}^n$ covers $sp(b_o)$. Set $U_{\hat{x}_i} = U_i$, $a_{\hat{x}_i} = a_i$ and $m_{\hat{x}_i} = m_i$, $i = 1,2,\ldots,n$. Let $a_o \varepsilon A$ be such that $\hat{a}_o \equiv 1$ on $\bigcup_{i=1}^n \overline{U}_i$; this a_o exists by the regularity of A. Now define

$$e_1 = a_1,\ e_2 = a_2(a_o - a_1), \cdots,\ e_n = a_n(a_o - a_1)\cdots(a_o - a_{n-1}).$$

Evidently, $e_i \varepsilon A$ for each $i = 1,2,\cdots,n$, and we have

$$\sum_{i=1}^n \hat{e}_i = 1 \text{ on } \bigcup_{i=1}^n U_i \text{ since for each } i,$$

$$\hat{e}_i = \hat{a}_i(1 - \hat{a}_1)\cdots(1 - \hat{a}_{i-1}) \text{ on } \bigcup_{i=1}^n U_i \text{ and}$$

$$\sum_{i=1}^n \hat{e}_i = 1 - (1 - \hat{a}_1)(1 - \hat{a}_2)\cdots(1 - \hat{a}_n) \text{ on } \bigcup_{i=1}^n \overline{U}_i.$$

Let $e = \sum_{i=1}^n e_i$. Now $e \varepsilon A$ and since $\hat{e} = 1$ on the nbhd. $\bigcup_{i=1}^n U_i$ of $sp(b_o)$, proposition 3.7 implies $e*b_o = b_o$. Then we have

$$b_o = e*b_o = \sum_{i=1}^n e_i*b_o$$

$$= e_1*b_o + e_2*b_o + \cdots + e_n*b_o$$

$$= a_1*b_o + (a_o - a_1)*(a_2*b_o) + \cdots + (a_o - a_1)\cdots(a_o - a_{n-1})*(a_n*b_o)$$

$$= m_1 + (a_o - a_1)*m_2 + \cdots + (a_o - a_1)\cdots(a_o - a_{n-1})*m_n \in M.$$

Hence, $a_\alpha * b \in M$ for each $a_\alpha \in \langle a_\alpha \rangle$ and we obtain $b \in M$ by the definition of $\langle a_\alpha \rangle$. Q.E.D.

Remark 5.3

The conclusion of Theorem 5.4 remains valid for $b \in M$ and M a τ-closed submodule of B if τ is either the β or $*$-topology since in these cases there is an approximate identity which consists of elements belonging to A_c.

Lemma 5.3. Let B be a Banach A-module. An element $b \in B$ belongs locally to a submodule M at all points not in the spectrum of b.

Proof: Suppose $\hat{x} \in sp(b)^c$ and $a \in A$ such that $\hat{a} \equiv 1$ in some relatively compact nbhd. of $\hat{x}$ with $\sigma(a) \cap sp(b) = \emptyset$. Now $sp(a*b) = \emptyset$ implies $a*b = 0$. Therefore, b belongs locally to M at $\hat{x}$. Q.E.D.

We proceed to a proposition which will enable us to reach our objective as well as being of independent interest. In fact, it appears to provide sufficient conditions for the validity of spectral synthesis. We remark that for Banach algebras, the relation "$cosp(I) \subseteq cosp(a) \Longrightarrow a \in I$" holds for a closed ideal I of A only if $cosp(I)$ is a set of synthesis. Theorem 4.1 entails that for τ-bi-annihilation invariant modules we have the analogous relation "$sp(b) \subseteq sp(M) \Longrightarrow b \in M$" holding for a τ-closed submodule M of B only if $sp(M)$ is an S-B set. The following proposition therefore does not guarantee spectral synthesis but does give conditions to insure that elements belong to a submodule.

Proposition 5.4. Let B be a Banach A-module satisfying HB2. Let M

be a τ-closed submodule of B. If $b \in B$ satisfies $sp(b) \subset int(sp(M))$, then $b \in M$.

<u>Proof</u>: Evidently, we may assume $b \neq 0$ and so $int(sp(M))$ is non-empty. Suppose $b \notin M$, then by <u>HB2</u>" there exists an element $a_o \in M^{\perp A} \setminus b^{\perp A}$. We assert that $\sigma(a_o) \cap int(sp(M))$ is empty. To show this, let $\hat{x} \in \sigma(a_o) \cap int(sp(M))$. Now $\sigma(a_o) = c\ell(Z(a_o)^c)$ implies the existence of a net $\langle \hat{x}_\alpha \rangle \subset Z(a_o)^c$ with $\hat{x}_\alpha \to \hat{x}$. But $int(sp(M))$ is a nbhd. of $\hat{x}$ and so $\langle \hat{x}_\alpha \rangle$ is eventually in it. Let $\hat{y}$ be any $\hat{x}_\alpha$ in $int(sp(M))$. Observe that $\hat{y} \in Z(a_o)^c \cap int(sp(M))$. Let U be any relatively compact nbhd. of $\hat{y}$ such that

$$\overline{U} \subset Z(a_o)^c \cap int(sp(M)).$$

Since $sp(M) = c\ell(\bigcup_{m \in M} sp(M))$, the regularity of A entails $\hat{y} \in Z(a_o)^c \cap (\bigcup_{m \in M} sp(m))$, so that there is a $b_1 \in M$ with $b_1 \neq 0$ and $\hat{y} \in sp(b_1)$. Let $a_1 \in A$ be such that $\sigma(a_1) \subset U$, $\hat{a}_1 \equiv 1$ on $\overline{V}$, V a nbhd. of $\hat{y}$ with $V \subset \overline{V} \subset U$, then $b_o \equiv a_1 * b_1 \in M$ and $\hat{y} \in sp(b_o)$. Since $a_o \in M^{\perp A}$, $a_o * b_o = 0$. Thus, $\hat{a}_o(\hat{y}) = 0$. But this is a contradiction since $\hat{y} \in Z(a_o)^c$. Hence, the assertion $\sigma(a_o) \cap int(sp(M)) = \phi$ holds. Now

$$sp(a_o * b) \subseteq \sigma(a_o) \cap sp(b) \subseteq \sigma(a_o) \cap int(sp(M)) = \phi,$$

so that $a_o * b = 0$ which contradicts the choice of a_o and hence $b \in M$. Q.E.D.

We now apply proposition 5.4 to obtain a companion lemma to 5.3.

<u>Lemma 5.5</u>. Let B be a Banach A-module satisfying <u>HB2</u>. Let M be a

τ-closed submodule of B. An element $b \in B$ belongs locally to M at every point not in p-cosp(M).

Proof: Suppose $\hat{x} \notin$ p-cosp(M), then by Lemma 5.3 we may assume $\hat{x} \in sp(b)$. Note that $\text{p-cosp}(M)^c = (\bigcap_{m\in M} \text{p-cosp}(m))^c = \bigcup_{m\in M} (\text{p-cosp}(M))^c$. But $\text{p-cosp}(m)^c = \Sigma(m)$ so that there exists an $m \in M$ with $\hat{x} \in \Sigma(m) \cap sp(b)$. Let W be a nbhd. of $\hat{x}$ with $W \subset \Sigma(m)$. Let U be a relatively compact nbhd. of $\hat{x}$ with $\overline{U} \subset W$ and $a \in A$ such that $\hat{a} \equiv 1$ on U and $\sigma(a) \subset W$. We therefore have

$$sp(a*b) \subseteq \sigma(a) \cap sp(b) \subseteq W \subset \Sigma(m) \subset int(sp(M)).$$

By proposition 5.4, $a*b \in M$. The definition of a entails that b belongs locally to M at $\hat{x}$.

Q.E.D.

Corollary 5.6. Let B be a Banach A-module satisfying HB2. A τ-closed submodule M contains every $b \in B$ with compact spectrum disjoint from p-cosp(M).

Proof: Let $b \in B$ have compact spectrum disjoint from p-cosp(M). Evidently, $sp(b) \subseteq (\text{p-cosp}(M))^c$. By lemma 5.5, b belongs locally to M at each point not in p-cosp(M) and hence at each point of $sp(b)$. Applying lemma 5.3, b belongs locally to M at each point not in $sp(b)$. Therefore, b belongs locally to M everywhere and $b \in M$.

Q.E.D.

§3. A Wiener-Ditkin-Shilov Theorem

We are now in a position to formulate and prove a Banach module analogue of the "Wiener-Ditkin-Shilov Theorem" for Banach algebras. The development up to this time has been somewhat standard with necessary modifications and utilization of the duality condition HB2. It appears as though the requirement of condition HB2 may not be weakened at the present time, but modules to which we apply our results satisfy this condition. We are therefore satisfied with our present formulation of (weak-) condition (D) and Wiener-Ditkin-Shilov Theorem. Our module version embodies the essential feature of the Banach algebra formulation, that is, the condition of topological simplicity.

Now to some preliminary results. Given $b \in B$, denote the set of all $\hat{x} \in \Delta(A)$ for which b does not belong locally to a submodule M at $\hat{x}$ by $P(b,M)$. By lemmas 5.3 and 5.5, we see that $P(b,M) \subseteq sp(b) \cap p\text{-}cosp(M)$ if B satisfies HB2.

Lemma 5.7. Let M be a submodule of B and $b \in B$. The set $P(b,M)$ is closed.

Proof: Let $\hat{x} \in P(b,M)^c$. Since b belongs locally to M at $\hat{x}$, there is a nbhd. U of $\hat{x}$ and an $a \in A$ such that $\hat{a} \equiv 1$ on U and $a*b \in M$. But then b belongs locally to M at each point of U, hence $U \subset P(b,M)^c$. Thus, $P(b,M)^c$ is open. Q.E.D.

Here is a useful result in our development of a Wiener-Ditkin-Shilov Theorem (compare [67, p. 172]).

Proposition 5.8. Let B be a Banach A-module p-satisfying condition (D). Suppose M is a τ-closed submodule and $b \in B$ with $P(b,M) \subseteq \partial(p\text{-}cosp(B))$, then $P(b,M)$ contains no isolated points.

Proof: Suppose $\hat{x} \in P(b,M)$ is isolated, then there is a nbhd. U of $\hat{x}$ such that b belongs locally to M at each $\hat{y} \in U \setminus \{\hat{x}\}$. Let $a \in A$ be such that $\hat{a} \equiv 1$ on a nbhd. V of $\hat{x}$ satisfying $\overline{V} \subset U$, $\sigma(a) \subset U$. Now $\hat{x} \in P(b,M) \subseteq \partial(p\text{-}cosp(b)) \subseteq p\text{-}cosp(b)$ and so by condition (D) there is a sequence $\langle a_n \rangle \subset A$ with $\hat{a}_n \equiv 0$ on a nbhd. W_n of $\hat{x}$, for each n, and $||a_n*b - b||_B \to 0$. Set $b_n = a_n*(a*b)$. We now show $b_n \in M$ for all n.

Suppose $\hat{y} \notin U$, then $\hat{y} \notin sp(b_n)$ because $sp(b_n) \subseteq \sigma(a_n) \subset U$. Thus, lemma 5.3 gives to give b_n belonging locally to M at $\hat{y}$. Since $sp(b_n)$ is compact, it only remains to show b_n belongs locally to M at each $\hat{z} \in U$. If $\hat{z} \neq \hat{x}$, then b belongs locally to M at $\hat{z}$ and it is evident then that b_n belongs locally to M at $\hat{z}$. Now consider $\hat{z} = \hat{x}$. We have $sp(b_n) \subseteq sp(a_n{*}b)$ and so $\Sigma(b_n) \subseteq sp(a_n{*}b)$. This inclusion renders $p\text{-}cosp(b_n) \supseteq sp(a_n{*}b)^c$. But the definition of $\langle a_n \rangle$ entails $\hat{x} \in sp(a_n{*}b)^c$, and hence lemma 5.3 applies to get $a_n{*}b$ belonging locally to M at $\hat{x}$. Thus, b_n belongs locally to M at $\hat{x}$ and consequently $b_n \in M$ since it belongs locally to M everywhere. But M is τ-closed and thus, in fact, closed. Since $||a_n{*}b - b||_B \to 0$, the relation

$$||b_n - a{*}b||_B = ||a_n{*}(a{*}b) - a{*}b||_B = ||a{*}(a_n{*}b) - a{*}b||_B$$

$$\leq ||a||_A||a_n{*}b - b||_B$$

entails $a{*}b \in M$. By the construction of a, b belongs locally to M at $\hat{x}$. Therefore, we have a contradiction to the assumption that $\hat{x}$ is isolated. Q.E.D.

As an application, we formulate a Banach module analogue of a Wiener-Ditkin-Shilov Theorem. In so doing, we provide sufficient conditions for spectral synthesis. Although it does not directly resemble the formulation for Banach algebras, our module version is, indeed, a plausible version of a Wiener-Ditkin-Shilov Theorem since it encompasses the essential features and spirit of such a theorem.

Theorem 5.9. Let B be a Banach A-module satisfying HB2 and p-satisfying condition (D). For each $b \in B$ with $sp(b) \subseteq sp(M)$ and $\partial sp(b)$ containing no non-empty perfect sets, $b \in M$.

Proof: Since $sp(b) \subseteq sp(M)$, $\Sigma(b) \subseteq int(sp(M))$. But $\Sigma(b)$ is open so that for any point $\hat{x} \in \Sigma(b)$, there is a nbhd. U of $\hat{x}$ and an $a \in A_c$ such that $\hat{a} \equiv 1$ on a nbhd. V of $\hat{x}$ where $\overline{V} \subset U$ and $\sigma(a) \subset U$. Now $sp(a*b) \subset U \subset int(sp(M))$. By proposition 5.4, $a*b \in M$ and so b belongs locally to each point of $\Sigma(b)$. For $\hat{x} \in sp(b)^c$, lemma 5.3 applies so that b belongs locally to M at $\hat{x}$. Hence, $P(b,M) \subseteq \Sigma(b)^c \cap sp(b) = p\text{-}cosp(b) \cap sp(b)$. But

$$p\text{-}cosp(b) \cap sp(b) = \Sigma(b)^c \cap [\Sigma(b) \cup \partial sp(b)] = \partial sp(b)$$

so that $P(b,M) \subseteq \partial sp(b)$. Proposition 5.8 renders $P(b,M)$ perfect and so the hypothesis entails $P(b,M)$ empty. Therefore, b belongs locally to M at every point of $\Delta(A)$. Since B satisfies condition (D) at infinity, there is a sequence $\langle a_n \rangle \subset A_c$ such that $||a_n*b - b||_B \to 0$. Evidently, $sp(a_n*b)$ is compact and so a_n*b belongs locally to M everywhere. Thus, $a_n*b \in M$ and since M is closed, $b \in M$. Q.E.D.

Corollary 5.10. Let B be a Banach A-module satisfying HB2 and condition (D). If E is a closed subset of $\Delta(A)$ with E containing no non-empty perfect sets, then E is a set of spectral synthesis for B.

Proof: Suppose M and N are any τ-closed submodules of B with $sp(M) = sp(N) = E$. Theorem 5.9 entails that for each $b \in N$, since $sp(b) \subseteq sp(N) = E$, $b \in M$, and similarly, $b \in M$ satisfies $sp(b) \subseteq sp(M)$ which implies $b \in N$. Hence, $M = N$ and E is an S-B set. Q.E.D.

Remark 5.10

It would be of interest to furnish a theorem more closely resembling the Banach algebra version of a Wiener-Ditkin-Shilov Theorem which requires $\text{p-cosp}(M) \subseteq \text{p-cosp}(b)$ and $\partial(\text{p-cosp}(b)) \cap \partial(\text{p-cosp}(M))$ to contain no nonempty perfect sets. However, our p-cospectrum while adequate to obtain the version given in Theorem 5.9 is not "good enough" to establish a theorem with these other conditions. The problem lies in the fact that one may not be able to establish whether or not an element $b \in B$ belongs locally to a submodule M at each point in $\text{int}(\text{p-cosp}(b))$ while lemma 5.3 does guarantee this for elements in $\text{sp}(b)^c$. Generally, $\text{int}(\text{p-cosp}(b)) \supseteq \text{sp}(b)^c$ which again points out a difficulty in recapturing the cospectrum from the spectrum.

In the event that $B \subseteq A$, Theorem 5.9 is not comparable to the Banach algebra version of Wiener-Ditkin-Shilov Theorem. For instance, not all Banach algebras may satisfy HB2 and so is in this sense weaker. However, if B satisfies condition (D) as an algebra it p-satisfies condition (D) so that in this sense our result is stronger. The relative strength of Theorem 5.9 occurs in particular cases and its availability to modules which are not Banach algebras. We shall pursue this at a later time.

We conclude this section by stating sufficient conditions for spectral synthesis in Banach modules which are easily verified by using duality and the Wiener-Ditkin-Shilov Theorem for Banach algebras. This will play an interesting role in sections §5-§6.

<u>Theorem 5.11</u>. Let A satisfy condition (D) and B be a Banach A-module satisfying <u>HB2</u>. If M is a closed submodule of B and $b \in B$ with $sp(b) \subseteq sp(M)$ and $\partial sp(b)$ containing no nonempty perfect sets, then $b \in M$.

<u>Proof</u>: Suppose $a \in M^{\perp}$, then $sp(b) \subseteq sp(M)$ entails $cosp(a) \supseteq hull(b^{\perp})$. By the Wiener-Ditkin-Shilov Theorem for A (cf. I, §3, Fact 8), since $\partial sp(b) = \partial hull(b^{\perp})$ contains no nonempty perfect sets, we have $a \in b^{\perp}$. Therefore, $M^{\perp} \subseteq b^{\perp}$ and condition <u>HB2</u> implies $b \in M$. Q.E.D.

§4. Spectra and Almost Periodicity

We will now be concerned with the concept of almost periodicity with our perspective in terms of spectra. Our treatment is to relate to spectral synthesis considerations of almost periodicity previously done, and, in addition, emphasize the role of spectra. Again, it is intended that our study bring forth justification for consideration of spectral synthesis and analysis in the context of Banach modules. In particular, we re-interpret part of Kitchen's treatment of almost periodicity and determine a relation between spectra and almost periodic elements.

Before embarking on our proposed program, we adopt a definition of almost periodicity due to Kitchen [69] and provide some elementary results regarding such elements. Furthermore, the role of bi-annihilation invariance will be of particular interest.

<u>Definition 5.4 (Kitchen [69])</u>. Let B be a Banach A-module. An element $b \in B$ is <u>almost periodic</u> if the map $T_b : a \to a*b$ from A to B is compact. The module B is <u>almost periodic</u> if each element

is almost periodic.

It is not difficult to see that this extends the usual notion of almost periodicity. Furthermore, the above definition is equivalent to: b is almost periodic if the set $\{a*b : ||a||_A \leq 1\}$ is relatively compact in B.

The maximal closed almost periodic submodule for a Banach A-module B will be denoted by $AP(B,A)$. The existence of such a submodule is provided by our next proposition.

Proposition 5.12. Let B be a Banach A-module. There is a maximal closed submodule $AP(B,A)$ which is almost periodic.

Proof: The proof is straightforward using the properties of compact operators and we only provide a brief sketch. Let $AP(B,A) = \{b \in B : b$ is almost periodic$\}$. Evidently $0 \in AP(B,A)$ and $AP(B,A)$ is a submodule since T_b compact implies T_{a*b} is compact for any $a \in A$. To see that $AP(B,A)$ is closed, let $\langle b_\alpha \rangle \subset AP(B,A)$ converge to $b \in B$ in the B-norm. Then

$$||T_b - T_{b_\alpha}|| = \sup_{||a||\leq 1} ||T_b(a) - T_{b_\alpha}(a)||_B = \sup_{||a||\leq 1} ||a*b-a*b_\alpha||_B$$

$$\leq \sup_{||a||\leq 1} ||a||_A ||b - b_\alpha||_B = ||b - b_\alpha||_B \to 0.$$

Hence, T_b as a uniform limit of compact operators is compact or equivalently, $b \in AP(B,A)$. Since $AP(B,A)$ is evidently maximal, the conclusion follows. Q.E.D.

Remark 5.12

Proposition 5.12 asserts the existence of $AP(B,A)$, but not non-triviality. In some cases, $AP(B,A)$ contains no non-trivial elements, i.e., $AP(B,A) = \{0\}$. For example, if G is a non-compact LCAG, then the $L^1(G)$-modules $L^p(G)$, $1 < p < \infty$, have a trivial almost periodic part, i.e., $AP(L^p(G), L^1(G)) = \{0\}$. This fact has at times been overlooked, for example, see [26].

We now state an elementary condition which provides for $AP(B,A)$ to be non-trivial. This is well known for the classical case $B = C(G)$, $A = L^1(G)$ and G compact.

Proposition 5.13. Let B be a Banach A-module. If $b \in B$ has one-point spectrum, then b is almost periodic.

Proof: Let $b \in B$ be such that $sp(b) = \{\hat{x}\}$. Clearly $b \neq 0$. We note that $T_b : A \to B$ has range equal to $A*b$. Since $sp(b) = \{\hat{x}\}$, theorem 4.9 applies to yield $\dim([b]) = 1$, but then this amounts to dim(range of T_b) = 1. But an operator with finite dimensional range must be compact and hence, $b \in AP(B,A)$. Q.E.D.

Kitchen is concerned with the following spectral analysis question: when is every closed submodule of $AP(B,A)$ decomposable into one-dimensional submodules? It is evident that a necessary condition for this is to be true is that B contain elements of one-point spectra, in which case a closed submodule is "some" direct sum of $J(\{\hat{x}\})^{\perp B}$, $\hat{x} \in \Delta(A)$. Thus, for non-compact G, the $L^1(G)$-modules, $L^p(G)$, $1 \leq p < \infty$, not containing any characters cannot satisfy the above question. In particular, $AP(L^p(G), L^1(G)) = \{0\}$ as mentioned in remark 5.12.

We now state a sufficient condition for a module to be almost periodic. As an interesting consequence, we find a sufficient condition for almost periodicity (Theorem 5.14 and Corollary 5.15).

Theorem 5.14. If B is a bi-annihilation invariant A-module, then $B = c\ell(\bigcup_{\hat{x}\varepsilon\Delta(A)} M_{\hat{x}})$, where $M_{\hat{x}} = J(\{\hat{x}\})^{\perp B}$ for each $\hat{x} \varepsilon \Delta(A)$.

Proof: Annihilating $\bigcup_{\hat{x}\varepsilon\Delta(A)} M_{\hat{x}}$, we obtain by proposition 3.23

$$[\bigcup_{\hat{x}\varepsilon\Delta(A)} M_{\hat{x}}]^{\perp} = (\bigcap_{\hat{x}\varepsilon\Delta(A)} M_{\hat{x}}^{\perp})^{\perp\perp} = \bigcap_{\hat{x}\varepsilon\Delta(A)} M_{\hat{x}}^{\perp} = \bigcap_{\hat{x}\varepsilon\Delta(A)} J(\{\hat{x}\})^{\perp\perp}.$$

Therefore, bi-annihilation invariance and semisimplicity entail $[\bigcup_{\hat{x}\varepsilon\Delta(A)} M_{\hat{x}}]^{\perp} = \bigcap_{\hat{x}\varepsilon\Delta(A)} J(\{\hat{x}\}) = \bigcap_{\hat{x}\varepsilon\Delta(A)} I_{\hat{x}} = \{0\}$. Hence, annihilating we see that $[\bigcup_{\hat{x}\varepsilon\Delta(A)} M_{\hat{x}}]^{\perp\perp} = B$. Appealing to bi-annihilation invariance once more, we have $B = c\ell(\bigcup_{\hat{x}\varepsilon\Delta(A)} M_{\hat{x}})$. Q.E.D.

Remarks 5.14

1. Recalling that the family $\{M_{\hat{x}} : \hat{x} \varepsilon \Delta(A)\}$ is linearly independent by proposition 4.12, we, in fact, have $B = c\ell(\sum_{\hat{x}\varepsilon\Delta(A)} M_{\hat{x}})$, the topological direct sum. Moreover, if $b \varepsilon M_{\hat{x}} \cap c\ell(\sum_{\hat{y}\neq\hat{x}} M_{\hat{y}})$, then $b \varepsilon c\ell(\sum_{\hat{y}\neq\hat{x}} M_{\hat{y}})$ entails $[b] \subseteq c\ell(\sum_{\hat{y}\neq\hat{x}} M_{\hat{y}})$. By theorem 4.1, $b \varepsilon M_{\hat{x}}$ entails $[b] = M_{\hat{x}}$ so that $M_{\hat{x}} \subseteq c\ell(\sum_{\hat{y}\neq\hat{x}} M_{\hat{y}})$. Thus, for $b \neq 0$, we would have $B = c\ell(\sum_{\hat{y}\neq\hat{x}} M_{\hat{y}})$. We can then write $B = c\ell(\sum_{\hat{y}\varepsilon\Delta_1} M_{\hat{y}})$ where $\Delta_1 = \{\hat{x} \varepsilon \Delta(A) : M_{\hat{x}} \cap c\ell(\sum_{\hat{y}\neq\hat{x}} M_{\hat{y}}) = \{0\}\}$. In this way, B is strongly almost periodic in the sense of [95].

2. Theorem 5.14 says that bi-annihilation invariant modules are decomposable (in the above sense) into one-dimensional submodules, thus

are almost periodic. The existence of elements with one-point spectra and Remark 4.9 (4) entail the existence of "τ-almost periodic" elements for τ-bi-annihilation invariant modules. We may consider "weak-almost periodicity" as an interpretation of the "τ-topology" and τ-bi-annihilation invariance. We hope to pursue this at a later time.

Here is a sufficient condition for an element of a Banach module to be almost periodic.

Corollary 5.15. Let B be a Banach A-module. If $b \in B$ and $[b]$ is a bi-annihilation invariant submodule, then $b \in AP(B,A)$.

Proof: By theorem 5.14, $[b] = AP([b],A)$. Since $[b]$ is closed in B, for $b' \in [b]$, $T_{b'} : A \to [b]$ is a compact operator from A into B, that is $AP([b],A) \subseteq AP(B,A)$. Hence, $b \in [b] \subseteq AP(B,A)$. Q.E.D.

Remark 5.15

Corollary 5.15 provides a "test" to see whether an element is almost periodic. It does not, however, assert that it simplifies any calculations, but does offer a different perspective with regard to almost periodicity.

If we consider $B = L^p(G)$, $A = L^1(G)$ for $1 \leq p < \infty$ and G a compact abelian group, then bi-annihilation invariance entails $AP(L^p(G), L^1(G)) = L^p(G)$. Furthermore, $AP(C(G), L^1(G)) = C(G)$. This indicates that for bi-annihilation invariant modules, $AP(B,A)$ is bi-annihilation invariant. A natural question is to ask for a larger class of modules for which $AP(B,A)$ is bi-annihilation invariant. Noting that for a noncompact LCAG G, $AP(L^\infty, L^1) = AP(G)$, it also would be of interest to determine whether $AP(G)$ is bi-annihilation invariant (recall that $L^\infty(G)$ is a $*$-bi-annihilation invariant $L^1(G)$-module). We hope to examine this at a later time.

§5. On a Characterization of Loomis

Two of our principle results follow to conclude this chapter. In this section we characterize almost periodic elements in terms of spectra in the spirit of Loomis' characterization for the almost periodic functions on a LCAG G. This requires methods employed in section §3, namely the concept of local membership together with condition (D) for Banach algebras. It is the author's contention that the similarity in methods is due to the fact that we are in an "appropriate setting." The very possibility of working in the module context emphasizes the roles played by the algebraic structure involved. In addition to the "classical" perspective, one gains insight into the problems of spectral synthesis and spectral analysis by regarding them in the module context as proposed in this thesis.

Theorem 5.15. Let A satisfy condition (D) and B be a Banach A-module satisfying HB2. If $b \in B$ has compact spectrum containing no nonempty perfect sets, then $b \in AP(B,A)$.

Proof: Applying lemma 5.3, $P(b,AP) \subseteq sp(b)$. Suppose $\hat{x} \in sp(b)$ with $Y \setminus \{\hat{x}\} \subset P(b,AP)^c$ for some nbhd. U of $\hat{x}$. Let $a_1 \in A_c$ be such that $\hat{a}_1 \equiv 1$ on a nbhd. V of $\hat{x}$, $\overline{V} \subset U$ and $\sigma(a_1) \subset U$. Consider $a_1 * b$. Then $sp(a_1*b) \subseteq \sigma(a_1) \subset U$ and a_1*b belongs locally to $AP(B,A)$ at all points of $\Delta(A)$ except possibly $\hat{x}$. Assert that $sp(a_1*b) \setminus \{\hat{x}\} \subseteq sp(AP(B,A))$. Suppose that there is a $\hat{y} \in sp(a_1*b) \setminus \{\hat{x}\}$ such that $\hat{y} \notin sp(AP(B,A))$. Let W be a nbhd. of $\hat{y}$ such that $W \cap sp(AP(B,A)) = \emptyset$. Now there is an $a \in A_c$ such that $\hat{a} \equiv 1$ on some nbhd. of $\hat{y}$, $\sigma(a) \subset W$ and $a*a_1*b \in AP(B,A)$. But then $sp(a*a_1*b) \subseteq sp(AP(B,A))$ because $a*a_1*b \in AP(B,A)$ and $sp(a*a_1*b) \subseteq \sigma(a) \subseteq W$ by the choice of a. This entails $a*a_1*b = 0$ and so $\hat{y} \in sp(a_1*b)$ implies $\hat{a}(\hat{y}) = 0$, a contradiction. Hence, $sp(a_1*b) \setminus \{\hat{x}\} \subseteq sp(AP(B,A))$.

If $\hat{x} \notin sp(AP(B,A))$, then since $sp(AP(B,A))$ is closed, $\hat{x}$ is an isolated point of $sp(a_1*b)$. In this case, let $a \in A$ be such that $\hat{a} \equiv 0$ on $sp(a_1*b) \setminus \{\hat{x}\}$ and $\hat{x} \in \sigma(a)$, then $sp(a*a_1*b) \subseteq \{\hat{x}\}$ entails $a*a_1*b \in AP(B,A)$. Hence, a_1*b belongs locally to $AP(B,A)$ at $\hat{x}$ and so a_1*b belongs locally to $AP(B,A)$ everywhere, i.e., $a_1*b \in AP(B,A)$. If $\hat{x} \in sp(AP(B,A))$, then $sp(a_1*b) \subseteq sp(AP(B,A))$. By hypothesis $sp(b)$ contains no nonempty perfect sets, and so, in particular, $sp(a_1*b)$ contains no nonempty perfect sets. Applying Theorem 5.11, we see that $a_1*b \in AP(B,A)$. Hence, in either case, $a_1*b \in AP(B,A)$.

The definition of a_1 entails b belongs locally to $AP(B,A)$ at $\hat{x}$. Therefore, $P(b,AP)$ contains no isolated points and so $P(b,AP)$ is a perfect subset of $sp(b)$. The hypothesis forces $P(b,AP)$ to be empty and so b belongs locally to $AP(B,A)$ everywhere, i.e., $b \in AP(B,A)$. Q.E.D.

An immediate consequence is the following.

Corollary 5.16. Let A satisfy condition (D) and B be a Banach A-module satisfying HB2. Suppose $b \in B$ has a spectrum containing no nonempty perfect sets, then $b \in AP(B,A)$.

Proof: Recalling that condition HB2 entails B is essential (proposition 3.25), there is a net $\langle a_\alpha \rangle \subset A_c$ such that $||a_\alpha*b - b||_B \to 0$.

But now each $a_\alpha * b$ satisfies the hypothesis of Theorem 5.15 so that $a_\alpha * b \in AP(B,A)$. Since $AP(B,A)$ is closed, $b \in AP(B,A)$. Q.E.D.

Example 3.18 exhibits that the $L^1(G)$-module $C_o(G)$ satisfies HB2. Essentially the same argument may be used to show that $C_u(G)$ satisfies HB2 (with respect to the norm topology). Armed with this fact and Theorem 5.15 we arrive at Loomis' characterization of almost periodic functions in terms of spectra. Recall that $AP(L^\infty(G), L^1(G)) = AP(G)$.

Corollary 5.17. Let $\psi \in L^\infty(G)$ have compact spectrum which contains no nonempty perfect sets, then ψ is almost periodic.

Proof: Since $sp(\psi)$ is compact there is a function $f \in L^1(G)$ satisfying $\hat{f} \equiv 1$ on a nbhd. of $sp(\psi)$. Thus, proposition 3.7 applies to give $\psi = f*\psi \in C_u(G)$. By I, §4, $L^1(G)$ satisfies condition (D), and since $C_u(G)$ satisfies HB2 theorem 4.16 renders $\psi \in AP(C_u(G), L^1(G)) = AP(G)$. Q.E.D.

Remarks 5.17

1. Corollary 5.16 indicates that Corollary 5.17 need not require $sp(\psi)$ compact, and hence, any bounded uniformly continuous function whose spectrum contains no nonempty perfect sets is almost periodic.

2. We refer to [82] for an example showing that Corollary 5.17 is "best possible" in the sense that a nonempty perfect compact set supports Borel measures whose inverse transform is not almost periodic.

3. A special case of Theorem 5.15 is Reiter's Lemma (cf., IV §3) as noted in [82] for $B = C_u(G)$ and $A = L^1(G)$. Accordingly, our feeling that Reiter's lemma should be true for modules is borne out although we do not have it in the form presented in IV, §3.

§6. A Theorem of Beurling

Our final result is a generalization to Banach modules of a theorem of Beurling concerning convolution equations. As an application of spectral considerations in Banach modules and Theorem 5.15, this result re-enforces our module-context contention. It is of interest to note that Loomis [82] cites the work of Reiter [90] and Lewitan [80], the latter dealing with almost periodic solutions of integral equations, but not Beurling's result. Beurling's theorem follows for modules as a consequence of Theorem 5.15.

Theorem 5.18. Let B be a Banach A-module satisfying HB2 and A satisfying condition (D). If $a \in A$ has $\mathrm{cosp}(a)$ containing no nonempty perfect sets, then for $b \in B$ the following are equivalent:

(i) $a*b = 0$;

(ii) $b \in AP(B,A)$ and $\mathrm{sp}(b) \subseteq \mathrm{cosp}(a)$.

Proof: (i)$\Longrightarrow$(ii) Suppose $a*b = 0$. Then evidently, $\mathrm{sp}(b) \subseteq \mathrm{cosp}(a)$. Let $\langle a_\alpha \rangle \subset A_c$ satisfy $||a_\alpha * b - b||_B \to 0$ (recall B satisfies HB2). Now $\mathrm{sp}(a_\alpha * b)$ is compact and $\mathrm{sp}(a_\alpha * b) \subseteq \sigma(a_\alpha) \cap \mathrm{sp}(b) \subseteq \sigma(a_\alpha) \cap \mathrm{cosp}(a)$. Therefore, $\mathrm{sp}(a_\alpha * b)$ cannot contain any nonempty perfect set and Theorem 5.15 applies to give $a_\alpha * b \in AP(B,A)$. Since $AP(B,A)$ is closed, the definition of $\langle a_\alpha \rangle$ entails $b \in AP(B,A)$.

(ii)$\Longrightarrow$(i) Suppose $b \in AP(B,A)$ and $\mathrm{sp}(b) \subseteq \mathrm{cosp}(a)$. Now $\mathrm{sp}(b) = \mathrm{hull}(b^\perp)$ entails $\mathrm{hull}(b^\perp) \subseteq \mathrm{cosp}(a)$. The hypothesis on $\mathrm{cosp}(a)$ and the fact that A satisfies condition (D) allows us to use the Wiener-Ditkin-Shilov Theorem for A to obtain $a \in b^\perp$, or $a*b = 0$.

Q.E.D.

Beurling's result in a special case!

<u>Corollary 5.19</u>. Let $f \in L^1(\mathbb{R})$. If $\hat{f}^{-1}(0)$ is countable with no finite limit point, then a function $\psi \in C_u(\mathbb{R})$ satisfies $f*\psi = 0$ if and only if ψ is almost periodic and $sp(\psi) \subseteq \hat{f}^{-1}(0)$.

<u>Proof</u>: This is immediate from 5.18 regarding $C_u(\mathbb{R})$ as an $L^1(\mathbb{R})$-module and noting that $cosp(f) = \hat{f}^{-1}(0)$ has no nonempty perfect sets by hypothesis.

Q.E.D.

The strength and advantages of "spectral" considerations in modules are apparent in our module formulation and results of this chapter, in particular, the results in §3 and theorems 5.15-5.19. We will again point out and recast our "contention" in the summary in the last chapter, as well as provide questions and a perspective of the problems encountered in our investigation.

CHAPTER VI

SUMMARY AND PROBLEMS

Our final chapter conveys the theme of our composition in two ways. First, a summary is given to briefly point out the degree of our success in presenting a spectral synthesis theory for Banach modules. Secondly, problems stemming from our study will be discussed to further substantiate the Banach module context as being an appropriate setting for spectra-related problems.

§1. A Summary

A primary concern of this thesis has been to exhibit that a spectral synthesis theory for Banach modules is not only plausible, but that the Banach module context is, indeed, an appropriate setting for such considerations. In fact, the theme of our work hinges on investigation of problems related to spectra in a Banach module setting.

A basis for a spectral synthesis theory was established in Chapter III. Basic properties of spectra have evidently been crucial throughout our investigation. In addition, a Spectral Synthesis Problem, Spectral Analysis Problem, and Closure Problem have been formulated. Study of these problems has been enhanced by the duality condition introduced: τ-bi-annihilation invariance. A brief look at some structural properties of such modules was also undertaken to gain more understanding of the concept.

Chapter IV contains a treatment of elementary spectral synthesis which is vital for a consistent spectral synthesis theory in modules. Moreover, consideration of elements with one-point spectra also substantiates our approach as we realize a unification of results in particular cases. Sets of spectral synthesis for Banach modules and Banach algebras are shown to be the same for τ-bi-annihilation invariant modules. In particular, this allows us to approach a clarification concerning such a "logical" equivalence for the case $B = C_o(G)$, $A = L^1(G)$ as queried by de Leeuw and Mirkil.

Standard techniques for spectral synthesis are developed for Banach modules in Chapter V. The concept of local membership is defined for modules and criteria for membership in submodules determined. An attempt is carried out to obtain a module formulation of a Wiener-Ditkin-Shilov Theorem. This entails a definition of cospectrum. While we are unable to recapture the complete "zero set" for an element in a Banach module, we do utilize a partial-cospectrum. Such a p-cospectrum allows one to define a "weak" form of condition (D). Even though we deal with a weak formulation of condition (D), a Banach module formulation of a Wiener-Ditkin-Shilov Theorem obtains. In particular, sufficient conditions for the validity of spectral synthesis are provided. Appearing to fall short of our goal in the determination of a Wiener-Ditkin-Shilov Theorem resembling that for Banach algebras, we do succeed in capturing the essence of such a theorem, namely, a condition (D) and the feature of topological simplicity. Hence, our venture does render support for our contention.

As a final contribution, we consider almost periodicity with regard to spectra. Sufficient conditions for elements to be almost periodic are provided and it is shown that bi-annihilation invariant modules are almost periodic. Furthermore, our study culminates in application of our concept of local membership merged with the condition HB2 and condition (D) for Banach algebras to furnish a characterization of almost periodic elements in the sense of Loomis. As a consequence, we obtain a result for modules concerning convolution equations in the spirit of Beurling. These results combined with our development of spectral synthesis considerations in modules serve as evidence to uphold the dominating contention of the thesis.

To underscore our theme, we proceed to discuss some problems arising from our study. The questions encountered provide additional motivation for regarding spectral synthesis problems in Banach modules. Many questions have naturally occurred in our investigation, however, we will mention only a few. Our selections are intended to accentuate our module perspective and reveal a unification of concepts as proposed in the thesis.

§2. Problems in a Definition of Cospectrum

Perhaps one of the most striking problems encountered in the thesis is the determination of a "good" definition of cospectrum for elements in Banach modules. The discussion in Chapter IV, §2, reveals that our partial-cospectrum does not extend the concept of zero set for elements in $A \cap B$. There may be alternate ways of defining the cospectrum of $b \in B$, say, in terms of cospectra of its "factors" in A in case $b = a_1 * b_1$ for some $a_1 \in A$, $b_1 \in B$ (cf., proposition 3.6). On the other hand, if a "maximal submodule space" were known, the cospectrum could be defined algebraically as in the concept of hull. Also, having a "generalized" Gelfand transform could lead to a more appropriate definition (we cite [70] for such a possibility). In any event, our study indicates that a "good" definition of cospectrum would allow a development of a Wiener-Ditkin-Shilov Theorem more closely resembling the Banach algebra version. With these notions in mind, we pose the following questions.

1. Does there exist a definition of cospectrum for Banach modules which extends the definition of cospectrum for elements in $A \cap B$?

2. Given a positive answer to question 1, under what conditions does a Banach A-module B satisfy condition (D) whenever A does, and conversely?

An answer for arbitrary (say essential) modules would extend the result in [15].

3. Given a positive answer to question 1, can one state and prove a Wiener-Ditkin-Shilov Theorem which extends that for Banach algebras?

4. In the case there is no "satisfactory" response to question 1, does this signify that Theorem 5.11 is best in the sense that a Wiener-Ditkin-Shilov Theorem for Banach modules necessarily entails a formulation in terms of spectrum?

§3. Questions on Thin Sets in Banach Modules

The problem encountered in defining the cospectrum and in extending Reiter's lemma (cf. IV, §2) appears to re-inforce the necessity for studying "thin sets." This is also hinted at in the consideration of the closure problem. Accordingly, it may be possible to capture the features of "independence" and arithmetic simplicity in the context of modules. We query:

Q. Can the concept of independence (as defined for groups) be introduced into maximal ideal spaces $\Delta(A)$ which may not necessarily possess any group structure?

We can repeat similar questions with regard to "thin sets." The problem appears to lie in a proper transition in interpretation in $\Delta(A)$ which may possess no group structure.

§4. Problems on Almost Periodicity

Our perspective to almost periodicity leads to a variety of possibilities in Banach modules. Coupled with previous investigations, our study reaffirms the notion that almost periodicity is inherent in spectral synthesis considerations. Of interest is the work [95] which gives rise to the following problems.

1. What is the relationship of strongly almost periodic (Heriditarily Strongly Almost Periodic) modules to bi-annihilation invariant modules?

2. Kitchen and Robbins [95] consider a spectral synthesis question in terms of maximal closed submodules. Unify our spectrum-oriented theory with that seeking to determine spectral synthesis with respect to a maximal closed submodule space.

The application of τ-bi-annihilation invariance in Theorem 5.17 reveals that τ-bi-annihilation invariant modules are τ-almost periodic, i.e., $B = c\ell_B^{\tau}(\bigsqcup_{\hat{x}\in\Delta(A)} M_x)$.

3. Investigate the concept of "weak-almost periodicity" in terms of τ-almost periodicity and τ-bi-annihilation invariance. In particular, interpret the use of the τ-topology with regard to τ-bi-annihilation invariance and almost periodicity.

In regard to sustaining our contention, there is a result due to de Vito [25] which characterizes closed ideals of $L^1(\mathbb{R})$ which are synthesizable. It is our feeling that a similar result is true for Banach modules. We pose a question.

4. Let I be a closed ideal of A and B be a Banach A-module. Under what conditions are the following two conditions equivalent?

(1) $I = [b]^{\perp_A}$ for some $b \in AP(B,A)$;

(2) hull(I) is an S-A set.

We suspect that at the very least HB2 is required. In fact, bi-annihilation invariance and condition (D) for A may be sufficient. The methods attempted at this time have been unsuccessful. In the event that the "norm spectrum" of an element $b \in AP(B,A)$ could be adequately defined, there may be additional hope in resolving the above problem, although this may not be necessary.

§5. Problems: Miscellany

We conclude with a few questions specifically arising from our investigation. A general question is whether the condition of τ-bi-annihilation invariance, HB1, or HB2 can be removed from the hypothesis of some of our results so that they obtain for a "larger" class of modules.

1. Does the conclusion of theorem 4.17 obtain for a larger class of modules? In particular, is a closed angular subsemigroup of $\hat{G}$ a

set of spectral synthesis for an arbitrary Banach module over a Banach algebra whose maximal ideal space is $\hat{G}$? (cf., lemma 4.16).

The considerations of the closure problem illustrates that our study provides a different perspective of formidable problems. It would be of interest to pursue the closure problems further in the Banach module context.

2. Does lemma 4.21 hold for a wider class of modules, say τ-bi-annihilation invariant modules?

3. With respect to the considerations in Chapter IV, §6, extend Theorem 4.24 and obtain other sufficient conditions for the closure property or decomposition property to hold.

As may be inferred from the problem in §2, other spectral synthesis related problems may be viewed in modules. A Wiener-Ditkin (or Calderon) set for a Banach algebra A is a closed set $E \subset \Delta(A)$ such that $a \in I(E)$ is in the closure of $J_o(E) \cdot a$. Along this vein, we ask the following.

4. Can the concept of a Wiener-Ditkin set be defined for Banach modules so as to obtain a meaningful theory?

5. In [61], the authors pose the question of determining which sets $E \subset \hat{G}$ satisfy $I(E) * I(E) = I(E)$. How does one recast this problem for Banach modules? Is the Banach module approach fruitful in characterizing such sets?

The need for a topology weaker than the norm topology is apparent in our study. A thorough investigation of the utilization of the strict topology awaits pursuit. For an A-Segal algebra B, application of the strict topology may shed light on the determination of the relative completion of B, $\tilde{B}^A$.

6. Regarding A-Segal algebras as Banach A-modules, what is the pertinence of the strict topology to questions regarding the relative completion?

As a final remark, we point out that our study does not employ any categorical techniques. While we have attempted to establish a spectral synthesis theory in Banach modules in order to acquire a more profitable perspective of the problem, we have but begun toward such a foundation. A natural query is whether categorical methods would nurture a spectral synthesis theory in certain categories of Banach modules. As mentioned in the introduction, the impact of algebra on analysis has been profound, our final question accentuates this fact.

7. How can one employ homological algebraic techniques to resolve spectral synthesis questions in categories of Banach or locally convex modules?

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